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A new approach to three-dimensional exact solutions for functionally graded piezoelectric laminated plates

G.M. Kulikov*, S.V. Plotnikova

Department of Applied Mathematics and Mechanics, Tambov State Technical University, Sovetskaya Street, 106, Tambov 392000, Russia

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ABSTRACT

A paper presents the sampling surfaces (SaS) method and its implementation for the three-dimensional (3D) exact analysis of functionally graded (FG) piezoelectric laminated plates. According to this method, we introduce inside the *n*th layer I_n not equally spaced SaS parallel to the middle surface of the plate and choose displacement vectors and electric potentials of these surfaces as basic plate variables. Such choice of unknowns with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction for each layer leads to a very compact form of governing equations of the FG piezoelectric plate formulation. This fact gives an opportunity to derive the 3D exact solutions of electroelasticity for thick and thin FG piezoelectric laminated plates with a specified accuracy utilizing a sufficient number of SaS, which are located at interfaces and Chebyshev polynomial nodes.

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1. Introduction

In recent years, a considerable work has been carried out on the three-dimensional (3D) exact analysis of piezoelectric laminated plates. In the literature, there are at least four approaches to 3D exact solutions of electroelasticity for piezoelectric laminated plates, namely, the Pagano approach, the state space approach, the asymptotic approach and the sampling surfaces (SaS) approach. The first approach [1,2] was applied to piezoelectric plates by Ray et al. [3], Heyliger [4,5], Heyliger and Brooks [6]. The 3D exact analysis of piezoelectric orthotropic and anisotropic plates based on the state space approach was carried out in contributions [7–11]. The asymptotic approach was utilized for derivation of 3D exact solutions for piezoelectric plates [12–15]. The SaS approach was recently implemented for the 3D exact analysis of piezoelectric and anisotropic plates [16].

Nowadays, the functionally graded (FG) piezoelectric materials are widely used in mechanical engineering due to their advantages compared to traditional piezoelectric laminated materials. However, the study of FG piezoelectric structures is not a simple task [17] because the material properties depend on the thickness coordinate and some specific assumptions concerning their variations in the thickness direction are required [18,19]. In practice, this implies that we deal here with a system of differential equations with variable coefficients. Therefore, the first two approaches, i.e., the Pagano approach and the state space approach cannot be applied directly to 3D exact solutions for FG piezoelectric plates without using above specific assumptions [20]. On the contrary, the asymptotic approach [21] and the SaS approach can be applied directly to 3D solutions for FG piezoelectric plates because governing differential equations are obtained through definite integration in the thickness direction.

The present paper is intended to show that the SaS method can be also applied efficiently to 3D exact solutions of electroelasticity for FG piezoelectric laminated plates. According to this method, we choose inside the *n*th layer I_n not equally spaced SaS $\Omega^{(n)1}, \Omega^{(n)2}, \ldots, \Omega^{(n)I_n}$ parallel to the middle surface of the plate and introduce the displacement vectors $\mathbf{u}^{(n)1}, \mathbf{u}^{(n)2}, \ldots, \mathbf{u}^{(n)I_n}$ and the electric potentials $\varphi^{(n)1}, \varphi^{(n)2}, \ldots, \varphi^{(n)I_n}$ of these surfaces as basic plate variables, where $I_n \ge 3$. Such choice of unknowns in conjunction with the use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction permits the presentation of governing equations of the proposed FG plate formulation in a very compact form. Note that the SaS method has been already applied to the 3D analysis of elastic and piezoelectric laminated plates and shells [16,22–25].

It should be mentioned that the developed approach with equally spaced SaS [22] does not work properly with Lagrange polynomials of high degree because the Runge's phenomenon can occur, which yields the wild oscillation at the edges of the interval when the user deals with any specific functions. If the number of equally spaced nodes is increased then the oscillations become even larger. Fortunately, the use of Chebyshev polynomial nodes [26] inside each layer can help to improve significantly the behavior of Lagrange polynomials of high degree for which the error will go to zero as $I_n \rightarrow \infty$.

An idea of using the SaS can be traced back to [27,28] in which three, four and five equally spaced SaS are employed. These





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^{*} Corresponding author. Tel.: +7 475 271 3299; fax: +7 075 271 0216. *E-mail address:* gmkulikov@mail.ru (G.M. Kulikov).

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contributions describe the SaS concept applied to the approximate solution of 3D plate/shell problems. For further information the reader refers to fundamental works [29,30] where the Legendre polynomials in the thickness direction are utilized. However, the use of Legendre polynomials cannot provide a *uniform convergence* of computational procedures to be developed. On the contrary, the SaS method leads to a uniform convergence, as shown in Section 5, that in turn gives an opportunity to derive the 3D exact solutions for FG piezoelectric laminated plates with a prescribed accuracy employing the sufficient number of SaS.

The authors restrict themselves to finding *five right digits* in all examples presented. The better accuracy is possible of course but requires more SaS inside the plate body to be taken.

2. Description of electric field

Consider a FG piezoelectric laminated plate of the thickness h. Let the middle surface Ω be described by Cartesian coordinates x_1 and x_2 . The coordinate x_3 is oriented in the thickness direction. The transverse coordinates of SaS inside the *n*th layer are defined as

$$\begin{aligned} x_{3}^{(n)1} &= x_{3}^{[n-1]}, \quad x_{3}^{(n)I_{n}} = x_{3}^{[n]}, \\ x_{3}^{(n)m_{n}} &= \frac{1}{2} \left(x_{3}^{[n-1]} + x_{3}^{[n]} \right) - \frac{1}{2} h_{n} \cos \left(\pi \frac{2m_{n} - 3}{2(I_{n} - 2)} \right), \end{aligned}$$
(1)

where $x_3^{[n-1]}$ and $x_3^{[n]}$ are the transverse coordinates of layer interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ (Fig. 1); $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the *n*th layer; the index *n* identifies the belonging of any quantity to the *n*th layer and runs from 1 to *N*, where *N* is the number of layers; the index m_n identifies the belonging of any quantity to inner SaS of the *n*th layer and runs from 2 to $I_n - 1$, whereas the indices i_n, j_n, k_n to be introduced later for describing all SaS of the *n*th layer run from 1 to I_n .

Remark 1. It is worth noting that transverse coordinates of inner SaS (1) coincide with coordinates of Chebyshev polynomial nodes [26]. This fact has a great meaning for a convergence of the SaS method [23–25].

The relation between the electric field vector and the electric potential φ is given by

$$E_i = -\varphi_i. \tag{2}$$

Here, and in the following developments, indices *i*, *j*, *k*, ℓ range from 1 to 3, whereas Greek indices α , β range from 1 to 2.

The electric field vector at SaS of the *n*th layer is written as



Fig. 1. Geometry of the laminated plate.

$$E_{\alpha}^{(n)i_{n}} = E_{\alpha}\left(x_{3}^{(n)i_{n}}\right) = -\varphi_{,\alpha}^{(n)i_{n}},\tag{3}$$

$$E_3^{(n)i_n} = E_3\left(x_3^{(n)i_n}\right) = -\psi^{(n)i_n},\tag{4}$$

where $\varphi^{(n)i_n}(x_1, x_2)$ are the electric potentials of SaS of the *n*th layer; $\psi^{(n)i_n}(x_1, x_2)$ are the values of the derivative of the electric potential with respect to thickness coordinate at SaS, that is,

$$\varphi^{(n)i_n} = \varphi\left(x_3^{(n)i_n}\right), \quad \psi^{(n)i_n} = \varphi_{,3}(x_3^{(n)i_n}).$$
 (5)

Next, we assume that the electric potential and the electric field vector are distributed through the thickness of the *n*th layer as follows:

$$\varphi^{(n)} = \sum_{i_n} L^{(n)i_n} \varphi^{(n)i_n}, \quad \mathbf{x}_3^{[n-1]} \leqslant \mathbf{x}_3 \leqslant \mathbf{x}_3^{[n]}, \tag{6}$$

$$E_i^{(n)} = \sum_{i_n} L^{(n)i_n} E_i^{(n)i_n}, \quad \mathbf{x}_3^{[n-1]} \leqslant \mathbf{x}_3 \leqslant \mathbf{x}_3^{[n]},$$
(7)

where $L^{(n)i_n}(x_3)$ are the Lagrange polynomials of degree $I_n - 1$ expressed as

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}}.$$
(8)

The use of (5) and (6) leads to a simple formula

$$\psi^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n})\varphi^{(n)j_n},\tag{9}$$

where $M^{(n)j_n} = L_3^{(n)j_n}$ are the derivatives of Lagrange polynomials, which are calculated at SaS of the *n*th layer as

$$\begin{split} M^{(n)j_n}\left(x_3^{(n)i_n}\right) &= \frac{1}{x_3^{(n)j_n} - x_3^{(n)i_n}} \prod_{k_n \neq i_n j_n} \frac{x_3^{(n)j_n} - x_3^{(n)k_n}}{x_3^{(n)j_n} - x_3^{(n)k_n}} \quad \text{for } j_n \neq i_n, \\ M^{(n)i_n}\left(x_3^{(n)i_n}\right) &= -\sum_{j_n \neq i_n} M^{(n)j_n}\left(x_3^{(n)i_n}\right). \end{split}$$
(10)

This implies that the key functions $\psi^{(n)i_n}$ of the electric field formulation are represented as a *linear combination* of electric potentials of SaS of the *n*th layer $\varphi^{(n)j_n}$.

3. Kinematic description of FG laminated plate

The strain tensor is given by

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i},\tag{11}$$

where u_i are the displacements of the plate. In particular, the strain components at SaS are

$$2\varepsilon_{\alpha\beta}^{(n)i_{n}} = 2\varepsilon_{\alpha\beta}\left(x_{3}^{(n)i_{n}}\right) = u_{\alpha\beta}^{(n)i_{n}} + u_{\beta\alpha}^{(n)i_{n}},$$

$$2\varepsilon_{\alpha3}^{(n)i_{n}} = 2\varepsilon_{\alpha3}\left(x_{3}^{(n)i_{n}}\right) = \beta_{\alpha}^{(n)i_{n}} + u_{3,\alpha}^{(n)i_{n}},$$

$$\varepsilon_{33}^{(n)i_{n}} = \varepsilon_{33}\left(x_{3}^{(n)i_{n}}\right) = \beta_{3}^{(n)i_{n}},$$
(12)

where $u_i^{(n)i_n}(x_1, x_2)$ are the displacements of SaS of the *n*th layer; $\beta_i^{(n)i_n}(x_1, x_2)$ are the values of derivatives of displacements with respect to coordinate x_3 at SaS, that is,

$$u_i^{(n)i_n} = u_i \left(x_3^{(n)i_n} \right), \quad \beta_i^{(n)i_n} = u_{i,3} \left(x_3^{(n)i_n} \right).$$
 (13)

The following step consists in a choice of consistent approximation of displacements and strains through the thickness of the *n*th layer. It is apparent that displacement and strain distributions should be chosen similar to electric field distributions (6) and (7):

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n} u_i^{(n)i_n}, \quad x_3^{[n-1]} \leqslant x_3 \leqslant x_3^{[n]},$$
(14)

$$\varepsilon_{ij}^{(n)} = \sum_{i_n} \mathcal{L}^{(n)i_n} \varepsilon_{ij}^{(n)i_n}, \quad \mathbf{x}_3^{[n-1]} \leqslant \mathbf{x}_3 \leqslant \mathbf{x}_3^{[n]}.$$
(15)

The use of (13) and (14) yields a formula

$$\beta_i^{(n)i_n} = \sum_{j_n} M^{(n)j_n} \left(\mathbf{x}_3^{(n)i_n} \right) u_i^{(n)j_n}, \tag{16}$$

which is similar to (9). Thus, the key functions $\beta_i^{(n)i_n}$ of the proposed laminated plate formulation are represented as a *linear combination* of displacements of SaS of the *n*th layer $u_i^{(n)j_n}$.

4. Variational formulation

The extended potential energy of the FG piezoelectric laminated plate [31] can be written as follows:

$$\Pi = \frac{1}{2} \int \int_{\Omega} \sum_{n} \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} \left(\sum_{i,j} \sigma_{ij}^{(n)} \varepsilon_{ij}^{(n)} - \sum_{i} D_{i}^{(n)} E_{i}^{(n)} \right) dx_{1} dx_{2} dx_{3} - W, \quad (17)$$

$$W = \int \int_{\Omega} \left[\sum_{i} (p_{i}^{+} u_{i}^{+} - p_{i}^{-} u_{i}^{-}) + q^{+} \varphi^{+} - q^{-} \varphi^{-} \right] dx_{1} dx_{2} + W_{\Sigma}, \quad (18)$$

where $\sigma_{ij}^{(n)}$ is the stress tensor of the *n*th layer; $D_i^{(n)}$ is the electric displacement vector of the *n*th layer; $u_i^- = u_i^{(1)1}$ and $u_i^+ = u_i^{(N)I_N}$ are the displacements of bottom and top surfaces; $\varphi^- = \varphi^{(1)1}$ and $\varphi^+ = \varphi^{(N)I_N}$ are the electric potentials of bottom and top surfaces; p_i^- and p_i^+ are the loads acting on outer surfaces; q^- and q^+ are the electric charges on outer surfaces; W_{Σ} is the work done by external electromechanical loads applied to the boundary surface Σ .

Substituting electric field and strain distributions (7) and (15) into functional (17) and introducing stress resultants

$$H_{ij}^{(n)i_n} = \int_{x_3^{[n-1]}}^{x_3^{(n)}} \sigma_{ij}^{(n)} L^{(n)i_n} dx_3,$$
(19)

and electric displacement resultants

$$T_i^{(n)i_n} = \int_{x_3^{[n-1]}}^{x_3^{[n]}} D_i^{(n)} L^{(n)i_n} \, dx_3, \tag{20}$$

one obtains

$$\Pi = \frac{1}{2} \int \int_{\Omega} \sum_{n} \sum_{i_n} \left(\sum_{i,j} H_{ij}^{(n)i_n} \varepsilon_{ij}^{(n)i_n} - \sum_{i} T_i^{(n)i_n} E_i^{(n)i_n} \right) dx_1 dx_2 - W.$$
(21)

For simplicity, we consider the case of linear piezoelectric materials [32] described as

$$\sigma_{ij}^{(n)} = \sum_{k,\ell} C_{ijk\ell}^{(n)} \varepsilon_{k\ell}^{(n)} - \sum_{k} e_{kij}^{(n)} E_{k}^{(n)}, \quad \mathbf{x}_{3}^{[n-1]} \leqslant \mathbf{x}_{3} \leqslant \mathbf{x}_{3}^{[n]},$$
(22)

$$D_{i}^{(n)} = \sum_{k,\ell} e_{ik\ell}^{(n)} \varepsilon_{k\ell}^{(n)} + \sum_{k} \epsilon_{ik}^{(n)} E_{k}^{(n)}, \quad \mathbf{x}_{3}^{[n-1]} \leqslant \mathbf{x}_{3} \leqslant \mathbf{x}_{3}^{[n]},$$
(23)

where $C_{ijkl}^{(n)}$, $e_{kij}^{(n)}$ and $\in_{ij}^{(n)}$ are the elastic, piezoelectric and dielectric constants of the *n*th layer.

Finally, we accept the last assumption of the FG piezoelectric plate formulation. Let us assume that material constants are distributed through the thickness of a plate according to the following law:

$$C_{ijkl}^{(n)} = \sum_{i_n} L^{(n)i_n} C_{ijkl}^{(n)i_n},$$
(24)

$$e_{kij}^{(n)} = \sum_{i_n} L^{(n)i_n} e_{kij}^{(n)i_n},$$
(25)

$$\in_{ik}^{(n)} = \sum_{i_n} L^{(n)i_n} \in_{ik}^{(n)i_n}$$
(26)

that is extensively utilized in this paper. Here, $C_{ijkl}^{(n)i_n}$, $e_{kij}^{(n)i_n}$ and $\in_{ik}^{(n)i_n}$ are the values of elastic, piezoelectric and dielectric constants on SaS of the *n*th layer.

Inserting constitutive Eqs. (22) and (23) respectively in Eqs. (19) and (20) and taking into consideration the through-thickness distributions (7), (15), (24), (25), and (26), we arrive at formulas for stress and electric displacement resultants:

$$H_{ij}^{(n)i_n} = \sum_{j_n,k_n} \Lambda^{(n)i_n j_n k_n} \left(\sum_{k,\ell} C_{ijk\ell}^{(n)j_n} \varepsilon_{k\ell}^{(n)k_n} - \sum_k e_{kij}^{(n)j_n} E_k^{(n)j_n} E_k^{(n)k_n} \right),$$
(27)

$$T_{i}^{(n)i_{n}} = \sum_{j_{n},k_{n}} \mathcal{A}^{(n)i_{n}j_{n}k_{n}} \left(\sum_{k,\ell} e_{ik\ell}^{(n)j_{n}} \varepsilon_{k\ell}^{(n)k_{n}} + \sum_{k} \in_{ik}^{(n)j_{n}} E_{k}^{(n)k_{n}} \right),$$
(28)

where

$$\Lambda^{(n)i_nj_nk_n} = \int_{x_3^{[n-1]}}^{x_3^{[n]}} L^{(n)i_n} L^{(n)j_n} L^{(n)k_n} \, dx_3.$$
⁽²⁹⁾

Now, the variational equation for the FG piezoelectric laminated plate in the case of conservative loading is written as

$$\delta \Pi = \mathbf{0}.\tag{30}$$

5. 3D exact solution for FG piezoelectric orthotropic plates

In this section, we study a simply supported FG piezoelectric laminated orthotropic rectangular plate. The edge boundary conditions of the plate are assumed to be fully supported and electrically grounded, that is,

$$\sigma_{11}^{(n)} = u_2^{(n)} = u_3^{(n)} = \varphi^{(n)} = 0 \quad \text{at } x_1 = 0 \text{ and } x_1 = a,$$

$$\sigma_{22}^{(n)} = u_1^{(n)} = u_3^{(n)} = \varphi^{(n)} = 0 \quad \text{at } x_2 = 0 \text{ and } x_2 = b,$$
(31)

where a and b are the plate dimensions. To satisfy boundary conditions, we search an analytical solution of the problem by a method of double Fourier series expansion

$$u_{1}^{(n)i_{n}} = \sum_{r,s} u_{1rs}^{(n)i_{n}} \cos \frac{r\pi x_{1}}{a} \sin \frac{s\pi x_{2}}{b}, \quad u_{2}^{(n)i_{n}} = \sum_{r,s} u_{2rs}^{(n)i_{n}} \sin \frac{r\pi x_{1}}{a} \cos \frac{s\pi x_{2}}{b},$$
$$u_{3}^{(n)i_{n}} = \sum_{r,s} u_{3rs}^{(n)i_{n}} \sin \frac{r\pi x_{1}}{a} \sin \frac{s\pi x_{2}}{b}, \quad \varphi^{(n)i_{n}} = \sum_{r,s} \varphi^{(n)i_{n}}_{rs} \sin \frac{r\pi x_{1}}{a} \sin \frac{s\pi x_{2}}{b},$$
(32)

where *r*, *s* are the wave numbers in plane directions. The external electromechanical loads are also expanded in double Fourier series.

Substituting (32) and Fourier series corresponding to electromechanical loading into the total potential energy (18) and (21) with $W_{\Sigma} = 0$ and allowing for (3), (4), (9), (12), (16), (27), and (28), one obtains

$$\Pi = \sum_{r,s} \Pi_{rs} \Big(u_{irs}^{(n)i_n}, \varphi_{rs}^{(n)i_n} \Big).$$
(33)

Invoking further the variational Eq. (30), we arrive at the system of linear algebraic equations

$$\frac{\partial \Pi_{rs}}{\partial u_{irs}^{(n)i_n}} = 0, \quad \frac{\partial \Pi_{rs}}{\partial \varphi_{rs}^{(n)i_n}} = 0 \tag{34}$$

of order $4(\sum_n l_n - N + 1)$. The linear system (34) can be easily solved by using a method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. Such a technique gives the

Table 1		
Elastic, piezoelectric and	dielectric properties	of materials.

Material	PZT-4	PZT-5A	Gr/Ep
C ₁₁₁₁ (GPa)	139.0	99.201	183.443
C ₂₂₂₂ (GPa)	139.0	99.201	11.662
C ₃₃₃₃ (GPa)	115.0	86.856	11.662
C ₁₁₂₂ (GPa)	77.8	54.016	4.363
C ₁₁₃₃ (GPa)	74.3	50.778	4.363
C ₂₂₃₃ (GPa)	74.3	50.778	3.918
C ₂₃₂₃ (GPa)	25.6	21.1	2.87
C ₁₃₁₃ (GPa)	25.6	21.1	7.17
C ₁₂₁₂ (GPa)	30.6	22.593	7.17
e_{311} (C/m ²)	-5.2	-7.209	0
e_{322} (C/m ²)	-5.2	-7.209	0
e_{333} (C/m ²)	15.1	15.118	0
e_{223} (C/m ²)	12.7	12.322	0
e_{113} (C/m ²)	12.7	12.322	0
\in_{11} (nF/m)	13.06	15.3	15.3
\in_{22} (nF/m)	13.06	15.3	15.3
\in_{33} (nF/m)	11.51	15.0	15.3

possibility to derive the exact solutions of 3D electroelasticity for FG piezoelectric orthotropic plates with a specified accuracy.

5.1. FG piezoelectric square plate

Consider a FG piezoelectric orthotropic square plate subjected to mechanical loading acting on its top surface

$$\sigma_{33}^{+} = -p_0 \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

$$\sigma_{13}^{-} = \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = D_3^{-} = D_3^{+} = 0$$
(35)

or electric loading acting on the bottom and top surfaces

$$D_{3}^{-} = D_{3}^{+} = q_{0} \sin \frac{\pi x_{1}}{a} \sin \frac{\pi x_{2}}{b},$$

$$\sigma_{13}^{-} = \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = \sigma_{33}^{+} = 0,$$
(36)

where $p_0 = 1$ Pa and $q_0 = 10^{-6} \text{ C/m}^2$.

Following [20] we assume that the FG material constants are distributed in the thickness direction according to the exponential law:

$$C_{ijkl} = C_{ijkl}^{-} e^{\alpha(z+0.5)}, \quad e_{ikl} = e_{ikl}^{-} e^{\alpha(z+0.5)}, \epsilon_{ik} = \epsilon_{ik}^{-} e^{\alpha(z+0.5)}, \quad z = x_3/h,$$
(37)

where C_{ijkl} , e_{ikl} and \in_{ik} are the values of elastic, piezoelectric and dielectric constants on the bottom surface, which are considered to be the same as those of the PZT-4 given in [20] and Table 1; α is the material gradient index, which can be determined as

$$\alpha = \ln \frac{C_{ijkl}^{+}}{C_{ijkl}^{-}} = \ln \frac{e_{ikl}^{+}}{e_{ikl}^{-}} = \ln \frac{\epsilon_{ik}^{+}}{\epsilon_{ik}^{-}},$$
(38)

where C^+_{ikl} , e^+_{ikl} and \in^+_{ik} are the values of elastic, piezoelectric and dielectric constants on the top surface.

To compare the derived results with an exact solution [20], we take a = b = 1 m and introduce the scaled field variables as follows:

Table 2

Results for a FG piezoelectric plate with a/h = 10 and $\alpha = -1$ under mechanical loading.

I_1	$\bar{u}_1(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$\bar{\sigma}_{13}(0)$	$\bar{\sigma}_{33}(0)$	$\overline{\textit{D}}_1(0.5)$	$\overline{D}_3(0)$
5	4.2738	-2.4856	-12.138	-7.8609	3.0219	-1.1787	-0.28738	0.91327	0.20612
7	4.2738	-2.4856	-12.138	-7.8544	3.0229	-1.1761	-0.28148	0.94503	0.21328
9	4.2738	-2.4856	-12.138	-7.8543	3.0229	-1.1762	-0.28150	0.94516	0.21323
11	4.2738	-2.4856	-12.138	-7.8543	3.0229	-1.1762	-0.28150	0.94516	0.21323

Table 3

Results for a FG piezoelectric plate with a/h = 10 and $\alpha = 1$ under mechanical loading.

I_1	$\bar{u}_1(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$ar{\sigma}_{13}(0)$	$ar{\sigma}_{ m 33}(0)$	$\overline{D}_1(0.5)$	$\overline{D}_3(0)$
5	1.1493	-0.92158	-4.4469	-15.336	6.0057	-1.1786	-0.21238	4.8146	-0.23971
7	1.1493	-0.92158	-4.4469	-15.360	6.0068	-1.1760	-0.21832	4.8517	-0.24667
9	1.1493	-0.92158	-4.4469	-15.360	6.0068	-1.1760	-0.21829	4.8519	-0.24662
11	1.1493	-0.92158	-4.4469	-15.360	6.0068	-1.1760	-0.21829	4.8519	-0.24662

Table 4

Results for a FG piezoelectric plate with a/h = 10 and $\alpha = -1$ under electric loading.

I_1	$\bar{u}_1(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$\bar{\sigma}_{13}(0)$	$\bar{\sigma}_{33}(0)$	$\overline{D}_1(0.5)$	$\overline{D}_3(0)$
5	208.76	-41.091	-200.62	182.37	147.65	0.18798	-0.22143	783.82	4854.2
7	208.76	-41.090	-200.68	182.80	147.66	0.14102	-0.25627	783.28	4856.0
9	208.76	-41.090	-200.68	182.80	147.66	0.14140	-0.25555	783.27	4856.0
11	208.76	-41.090	-200.68	182.80	147.66	0.14140	-0.25556	783.27	4856.0
13	208.76	-41.090	-200.68	182.80	147.66	0.14140	-0.25556	783.27	4856.0

Table 5 Results for a FG piezoelectric plate with a/h = 10 and $\alpha = 1$ under electric loading.

I_1	$\bar{u}_1(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$ar{\sigma}_{13}(0)$	$ar{\sigma}_{33}(0)$	$\overline{D}_1(0.5)$	$\overline{D}_3(0)$
5	27.457	15.116	73.803	192.64	143.51	-0.18798	-0.22143	1086.0	4854.2
7	27.457	15.116	73.827	193.26	143.50	-0.14102	-0.25627	1086.7	4856.0
9	27.457	15.116	73.827	193.26	143.50	-0.14140	-0.25555	1086.7	4856.0
11	27.457	15.116	73.827	193.26	143.50	-0.14140	-0.25556	1086.7	4856.0
13	27.457	15.116	73.827	193.26	143.50	-0.14140	-0.25556	1086.7	4856.0



Fig. 2. Distributions of transverse stresses, electric potential and electric displacement through the thickness of the FG piezoelectric square plate under mechanical loading for $I_1 = 9$: present analysis (–) and Zhong and Shang (\bigcirc).



Fig. 3. Distributions of transverse stresses, electric potential and electric displacement through the thickness of the FG piezoelectric square plate under electric loading for $I_1 = 9$: present analysis (-) and Zhong and Shang (\bigcirc).



Fig. 4. Accuracy of satisfying the boundary conditions δ_1^- , δ_3^- , δ_4^- and δ_1^+ , δ_3^+ , δ_4^+ on the bottom (\bigcirc) and top (\square) surfaces of the FG piezoelectric square plate with a/h = 2 and $\alpha = \pm 1$ under mechanical loading.



Fig. 5. Accuracy of satisfying the boundary conditions δ_1^- , δ_3^- , δ_4^- and δ_1^+ , δ_3^+ , δ_4^+ on the bottom (\bigcirc) and top (\square) surfaces of the FG piezoelectric square plate with a/h = 2 and $\alpha = \pm 1$ under electric loading.



Fig. 6. Distribution of elastic constants through the thickness of the FG rectangular plate.

Table 6 Results for a FG rectangular plate with $\beta = 2$ under mechanical loading.

$$\begin{split} \bar{u}_1 &= 10^{11} \times u_1(P,z), \quad \bar{u}_3 &= 10^{10} \times u_3(P,z), \quad \bar{\varphi} = 10^3 \times \varphi(P,z), \\ \bar{\sigma}_{11} &= \sigma_{11}(P,z), \quad \bar{\sigma}_{12} = \sigma_{12}(P,z), \quad \bar{\sigma}_{13} = \sigma_{13}(P,z), \quad \bar{\sigma}_{33} = \sigma_{33}(P,z), \\ \overline{D}_1 &= 10^{10} \times D_1(P,z), \quad \overline{D}_3 = 10^{10} \times D_3(P,z), \quad z = x_3/h, \end{split}$$

where P(a|4, a|4) is the point belonging to a middle surface. The data listed in Tables 2–5 for both cases of loading (35) and (36) of the FG piezoelectric square plate with a/h = 10 and $\alpha = \pm 1$ show that the SaS method permits the derivation of the 3D exact solution with a prescribed accuracy through the sufficient number of SaS. The first rows of these tables $(I_1 = 5)$ correspond to the Kulikov and Carrera fourth-order ESL formulation [28] with equally spaced SaS. It is seen that a choice of five SaS is sufficient to obtain the accurate results. However, in a proposed plate formulation the bottom and top surfaces are not included into a set of SaS because we deal here with a single-layer plate in which all SaS are located at Chebyshev polynomial nodes. Such an improvement leads to a uniform convergence. Figs. 2 and 3 display the distributions of transverse stresses, electric potential and electric displacement in the thickness direction for different values of the slenderness ratio a/h employing nine SaS. These results demonstrate convincingly the high potential of the

I ₁	$\bar{u}_1(0.5)$	$\bar{u}_2(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{22}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$\bar{\sigma}_{13}(0)$	$\bar{\sigma}_{23}(0)$	$\bar{\sigma}_{33}(0)$	$\overline{D}_1(0.5)$	$\overline{D}_3(0)$
5	-4.7884	-2.3942	3.6300	1.0529	54.286	33.567	-13.812	394.03	197.02	43.220	-11.151	3.6944
7	-4.7885	-2.3942	3.6630	1.0529	54.281	33.566	-13.810	390.87	195.43	43.429	-11.559	3.7001
9	-4.7885	-2.3942	3.6630	1.0529	54.280	33.565	-13.810	390.36	195.18	43.511	-11.506	3.7000
11	-4.7885	-2.3942	3.6630	1.0529	54.281	33.566	-13.810	390.79	195.39	43.441	-11.568	3.7020
13	-4.7885	-2.3942	3.6630	1.0529	54.281	33.566	-13.810	390.80	195.40	43.439	-11.570	3.7016
15	-4.7885	-2.3942	3.6630	1.0529	54.281	33.566	-13.810	390.80	195.40	43.439	-11.570	3.7017

proposed FG piezoelectric plate formulation. This is due to the fact that boundary conditions on the bottom and top surfaces of the shell for transverse components of the stress tensor and electric displacement vector are satisfied exactly by using the constitutive Eqs. (22) and (23). Additionally, Figs. 4 and 5 present the logarithmic errors

Га	b	le	7	

Results for a FG rectangular plate with $\beta = 2$ under electric loading.

I_1	$\bar{u}_1(0.5)$	$\bar{u}_2(0.5)$	$\bar{u}_3(0)$	$ar{arphi}(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{22}(0.5)$	$\bar{\sigma}_{12}(0.5)$	$\bar{\sigma}_{13}(0)$	$ar{\sigma}_{23}(0)$	$ar{\sigma}_{33}(0)$	$\overline{\textit{D}}_1(0.5)$	$\overline{D}_3(0)$
5	8.4568	4.2293	6.1634	-177.97	16.009	52.603	24.395	-3.5102	-1.7551	-0.57786	3383.6	3871.2
7	8.4568	4.2293	6.1633	-177.97	15.900	52.491	24.394	-3.6778	-1.8389	-0.67717	3383.5	3862.0
9	8.4568	4.2293	6.1633	-177.97	15.855	52.447	24.394	-3.6433	-1.8216	-0.67785	3383.5	3864.0
11	8.4568	4.2293	6.1633	-177.97	15.846	52.438	24.394	-3.6496	-1.8248	-0.67710	3383.5	3863.7
13	8.4568	4.2293	6.1633	-177.97	15.845	52.436	24.394	-3.6484	-1.8242	-0.67723	3383.5	3863.7
15	8.4568	4.2293	6.1633	-177.97	15.844	52.436	24.394	-3.6486	-1.8243	-0.67721	3383.5	3863.7



Fig. 7. Distributions of displacements, stresses, electric potential and electric displacement through the thickness of the FG rectangular plate under mechanical loading for $I_1 = 13$.



Fig. 8. Distributions of displacements, stresses, electric potential and electric displacement through the thickness of the FG rectangular plate under electric loading for $I_1 = 13$.

$$\begin{split} \delta_{1}^{-} &= lg \, |\bar{\sigma}_{13}(-0.5)|, \quad \delta_{1}^{+} = lg \, |\bar{\sigma}_{13}(0.5)|, \\ \delta_{3}^{-} &= lg \, |\bar{\sigma}_{33}(-0.5)|, \quad \delta_{3}^{+} = lg \, |\bar{\sigma}_{33}(0.5) + p_{0}/2|, \\ \delta_{4}^{-} &= lg \, 10^{8} \times |10^{-10} \times \overline{D}_{3}(-0.5) - q_{0}/2|, \\ \delta_{4}^{+} &= lg \, 10^{8} \times |10^{-10} \times \overline{D}_{3}(0.5) - q_{0}/2|, \end{split}$$
(40)

which help to assess the accuracy of fulfilling the boundary conditions for transverse stresses and electric displacement on outer surfaces of a plate. The results shown in Figs. 4 and 5 correspond respectively to the cases of $q_0 = 0$ and $p_0 = 0$. Note that the proposed SaS method provides a monotonic convergence that is impossible with *equally spaced* SaS [28].

Table 8Results for a FG piezoelectric angle-ply plate with $\beta = -2$ under mechanical loading.

In	$\bar{u}_1(0.5)$	$\bar{u}_3(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{13}(0)$	$\bar{\sigma}_{23}(0.25)$	$\bar{\sigma}_{33}(0)$	$ar{arphi}(-0.5)$	$\overline{D}_3(0.5)$
3	-2.8806	3.4829	5.0193	0.41940	-0.77314	0.50907	-0.43709	-0.30296
5	-2.8810	3.4927	5.0107	0.43015	-0.85600	0.50978	-0.43713	-0.30556
7	-2.8810	3.4927	5.0107	0.43013	-0.85585	0.50978	-0.43713	-0.30555
9	-2.8810	3.4927	5.0107	0.43013	-0.85585	0.50978	-0.43713	-0.30555

5.2. FG piezoelectric rectangular plate

Next, we study a FG piezoelectric orthotropic rectangular plate subjected to mechanical loading acting on the top surface

$$\sigma_{33}^{+} = p_0 \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

$$\sigma_{13}^{-} = \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = D_3^{-} = D_3^{+} = 0$$
(41)

or electric loading acting on the same surface

$$D_{3}^{+} = q_{0} \sin \frac{\pi x_{1}}{a} \sin \frac{\pi x_{2}}{b},$$

$$\sigma_{13}^{-} = \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = \sigma_{33}^{+} = D_{3}^{-} = 0,$$
(42)

where $p_0 = 1$ Pa and $q_0 = 10^{-7}$ C/m².

Here, we consider and compare two basic approaches widely used for describing the FG piezoelectric materials, namely, the exponential law (37) and the most popular power law [32]. The latter law reflects a simple rule of mixtures efficiently utilized for finding the effective properties of the FG piezoelectric material and can be presented as follows:

$$C_{ijkl} = C_{ijkl}^{-} V^{-} + C_{ijkl}^{+} V^{+}, \quad e_{ikl} = e_{ikl}^{-} V^{-} + e_{ikl}^{+} V^{+}, \in_{ik} = \in_{ik}^{-} V^{-} + \in_{ik}^{+} V^{+}, \quad V^{+} = 1 - V^{-},$$
(43)

where C_{ijkl}^- , e_{ik}^- , \in_{ik}^+ and C_{ijkl}^+ , e_{ik}^+ , e_{ik}^+ are the values of elastic, piezoelectric and dielectric constants on the bottom and top surfaces; $V^-(z)$ is the volume fraction given by

$$V^{-} = (0.5 - z)^{\beta}, \quad z = x_3/h,$$
 (44)

where β is the material gradient index.

The material constants on the bottom surface are considered to be the same as those of the PZT-4 given in Table 1, whereas the material constants on the top surface are three times more than those of the PZT-4. To investigate the response of the FG piezoelectric rectangular plate more carefully, we consider four values of the material gradient index: $\alpha = 1.0986$ in the case of using the exponential law (37), i.e. only one value can be chosen according to (38), and $\beta = 0.2, 2, 5$ in the case of the power law (43), which allows many values to be taken as illustrated in Fig. 6.

Table 9

Results for a FG piezoelectric angle-ply plate with $\beta = 2$ under mechanical loading.

The geometric parameters of the plate are taken as a = 1 m, b = 2 m and h = 0.1 m. To analyze the derived results for both types of loading (41) and (42) effectively, we introduce the following scaled field variables at crucial points:

$$\begin{split} \bar{u}_{1} &= 10^{11} \times u_{1}(0, b/2, z), \quad \bar{u}_{2} = 10^{11} \times u_{2}(a/2, 0, z), \\ \bar{u}_{3} &= 10^{10} \times u_{3}(a/2, b/2, z), \\ \bar{\sigma}_{11} &= \sigma_{11}(a/2, b/2, z), \quad \bar{\sigma}_{22} = \sigma_{22}(a/2, b/2, z), \quad \bar{\sigma}_{12} = \sigma_{12}(0, 0, z), \\ \bar{\sigma}_{13} &= 10^{2} \times \sigma_{13}(0, b/2, z), \quad \bar{\sigma}_{23} = 10^{2} \times \sigma_{23}(a/2, 0, z), \\ \bar{\sigma}_{33} &= 10^{2} \times \sigma_{33}(a/2, b/2, z), \\ \bar{\phi} &= 10^{2} \times \phi(a/2, b/2, z), \quad \overline{D}_{1} = 10^{10} \times D_{1}(0, b/2, z), \\ \bar{D}_{3} &= 10^{11} \times D_{3}(a/2, b/2, z). \end{split}$$
(45)

Tables 6 and 7 demonstrate again the high potential of the SaS method that yields the exact solution of 3D electroelasticity for FG piezoelectric rectangular plates with a prescribed accuracy using the sufficient number of SaS. The first rows of these tables ($I_1 = 5$) correspond to the fourth-order ESL formulation [28] with equally spaced SaS. However, the accuracy of computations is slightly worse than in the case of FG piezoelectric square plates. Figs. 7 and 8 present the distributions of displacements, stresses, electric potential and electric displacement through the thickness of the plate employing 13 SaS. As can be seen, the boundary conditions on the bottom and top surfaces for transverse components of the stress tensor and electric displacement vector are satisfied with a high accuracy.

6. 3D exact solution for FG piezoelectric anisotropic plates in cylindrical bending

Herein, we study a simply supported FG piezoelectric laminated anisotropic plate in cylindrical bending. The boundary conditions of the plate with electrically grounded edges are taken as

$$\sigma_{11}^{(n)} = \sigma_{12}^{(n)} = u_3^{(n)} = \varphi^{(n)} = 0 \quad \text{at } x_1 = 0 \text{ and } x_1 = a \tag{46}$$

In	$\bar{u}_{1}(0.5)$	$\bar{u}_3(0)$	$\bar{\sigma}_{11}(0.5)$	$ar{\sigma}_{13}(0)$	$ar{\sigma}_{23}(0.25)$	$ar{\sigma}_{33}(0)$	$ar{arphi}(-0.5)$	$\overline{D}_3(0.5)$
3	-1.8647	2.5297	11.045	0.37959	-0.48188	0.50864	-0.27007	-0.33897
5	-1.8649	2.5388	10.993	0.38675	-0.53847	0.50932	-0.27012	-0.35467
7	-1.8649	2.5388	10.993	0.38674	-0.53837	0.50932	-0.27012	-0.35487
9	-1.8649	2.5388	10.993	0.38674	-0.53837	0.50932	-0.27012	-0.35487

Table 10

Results for a FG piezoelectric angle-ply plate with $\beta = -2$ under electric loading.

In	$\tilde{u}_{1}(0.5)$	$\tilde{u}_3(0)$	$ ilde{\sigma}_{11}(0.5)$	$ ilde{\sigma}_{13}(0)$	$ ilde{\sigma}_{23}(0.25)$	$ ilde{\sigma}_{33}(0)$	$ ilde{arphi}(-0.5)$	$\tilde{D}_3(0.5)$
3	-5.3941	3.3518	-1.5376	0.9069	-1.4637	2.6756	-0.028463	-0.77512
5	-5.3982	3.3522	-1.5601	1.0061	-1.5662	2.8278	-0.028484	-0.78198
7	-5.3982	3.3522	-1.5600	1.0059	-1.5660	2.8276	-0.028484	-0.78195
9	-5.3982	3.3522	-1.5600	1.0059	-1.5660	2.8276	-0.028484	-0.78196
11	-5.3982	3.3522	-1.5600	1.0059	-1.5660	2.8276	-0.028484	-0.78196

Table 11 Results for a FG piezoelectric angle-ply plate with β = 2 under electric loading.

In	$\tilde{u}_1(0.5)$	$\tilde{u}_3(0)$	$ ilde{\sigma}_{11}(0.5)$	$ ilde{\sigma}_{13}(0)$	$ ilde{\sigma}_{23}(0.25)$	$ ilde{\sigma}_{33}(0)$	$ ilde{arphi}(-0.5)$	$\tilde{D}_3(0.5)$
3	-6.7380	3.8815	-0.41702	0.9449	-1.6615	3.7523	-0.024613	-1.3991
5	-6.7363	3.8769	-0.62918	1.0600	-1.7826	3.9646	-0.024614	-1.4614
7	-6.7363	3.8769	-0.63167	1.0598	-1.7823	3.9643	-0.024614	-1.4621
9	-6.7363	3.8769	-0.63172	1.0598	-1.7823	3.9643	-0.024614	-1.4621
11	-6.7363	3.8769	-0.63172	1.0598	-1.7823	3.9643	-0.024614	-1.4621

to simulate simple supports, where *a* is the width of the plate. In the case of the monoclinic piezoelectric material with a poling direction coincident with the x_3 axis, we can search an analytical solution of the problem as follows:

$$u_{1}^{(n)i_{n}} = \sum_{r=1}^{\infty} u_{1r}^{(n)i_{n}} \cos \frac{r\pi x_{1}}{a}, \quad u_{2}^{(n)i_{n}} = \sum_{r=1}^{\infty} u_{2r}^{(n)i_{n}} \cos \frac{r\pi x_{1}}{a}, u_{3}^{(n)i_{n}} = \sum_{r=1}^{\infty} u_{3r}^{(n)i_{n}} \sin \frac{r\pi x_{1}}{a}, \quad \varphi^{(n)i_{n}} = \sum_{r=1}^{\infty} \varphi_{r}^{(n)i_{n}} \sin \frac{r\pi x_{1}}{a}.$$
(47)

The external electromechanical loads are also expanded in Fourier series.

Substituting (47) and Fourier series corresponding to electromechanical loading into the extended potential energy (18) and (21) and taking into consideration Eqs. (3), (4), (9), (12), (16), (27), and (28), we obtain

$$\Pi = \sum_{r=1}^{\infty} \Pi_r(u_{ir}^{(n)i_n}, \varphi_r^{(n)i_n}).$$
(48)

The use of Eqs. (30) and (48) leads to a system of linear algebraic equations

$$\frac{\partial \Pi_r}{\partial u_{ir}^{(n)i_n}} = 0, \quad \frac{\partial \Pi_r}{\partial \varphi_r^{(n)i_n}} = 0$$
(49)

of order $4(\sum_n I_n - N + 1)$. The linear system (49) can be solved by a method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into



Fig. 9. Distributions of displacements, transverse stresses and electric displacement in the thickness direction of the FG piezoelectric angle-ply plate under mechanical loading for $I_1 = I_2 = I_3 = I_4 = I_5 = 9$: present analysis (-) and authors' 3D exact solution [16] (\bigcirc).

the numeric environment of MATLAB. That allows one to derive the 3D exact solutions of electroelasticity for FG piezoelectric laminated anisotropic plates in cylindrical bending with a specified accuracy.

As a numerical example, we study a symmetric three-ply plate with the stacking sequence [45/-45/45] made of the graphite-epoxy composite and covered with FG piezoelectric layers of equal thicknesses on its bottom and top surfaces [9]. This means that a five-ply plate with the stacking sequence [PZT/45/-45/45/PZT] is considered. The ply thicknesses are taken as $h_1 = h_5 = h/8$ and $h_2 = h_3 = h_4 = h/4$. The interfaces between the substrate and piezoelectric layers are electroded and grounded.

The material properties of the graphite–epoxy composite are given in Table 1. Concerning both FG piezoelectric layers it is

assumed that their material properties are distributed in the thickness direction according to a power law, that is

$$\begin{split} C_{ijkl}^{(1)} &= C_{ijkl}^{0} V_1(z), \quad e_{ikl}^{(1)} = e_{ikl}^{0} V_1(z), \\ &\in_{ik}^{(1)} = \in_{ik}^{0} V_1(z), \quad V_1(z) = (-8z/3)^{\beta}, \quad -1/2 \leqslant z \leqslant -3/8, \\ C_{ijkl}^{(5)} &= C_{ijkl}^{0} V_5(z), \quad e_{ikl}^{(5)} = e_{ikl}^{0} V_5(z), \\ &\in_{ik}^{(5)} = \in_{ik}^{0} V_5(z), \quad V_5(z) = (8z/3)^{\beta}, \quad 3/8 \leqslant z \leqslant 1/2, \end{split}$$
(50)

where C_{ijkl}^0 , e_{ik}^0 and \in_{ik}^0 are the elastic, piezoelectric and dielectric constants at interfaces between piezoelectric and substrate layers, which are considered to be the same as those of the PZT-5A given in [9] and Table 1; β is the material gradient index; $z = x_3/h$ is the dimensionless thickness coordinate. To investigate a response of the FG piezoelectric angle-ply plate more carefully, we consider five



Fig. 10. Distributions of displacements, transverse stresses and electric displacement in the thickness direction of the FG piezoelectric angle-ply plate under electric loading for $I_1 = I_2 = I_3 = I_4 = I_5 = 9$: present analysis (-) and authors' 3D exact solution [16] (\bigcirc).



Fig. 11. Accuracy of satisfying the boundary conditions δ_i^- , δ_4^- and δ_i^+ on the bottom (\bigcirc) and top (\square) surfaces of the FG piezoelectric angle-ply plate under mechanical loading: (a) $\beta = 0$ and (b) $\beta = 2$.

values of the material index $\beta = -2, -1, 0, 1, 2$. The case of $\beta = 0$ corresponds to an angle-ply plate with homogeneous piezoelectric layers. This allows a comparison with the authors' 3D exact solution [16].

The plate is subjected to mechanical loading acting on the top surface

$$\sigma_{33}^{+} = p_0 \sin \frac{\pi x_1}{a}, \sigma_{13}^{-} = \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = D_3^{-} = \varphi^{+} = 0$$
(51)

or electric loading acting on the same surface

$$\begin{aligned} \varphi^{+} &= \varphi_{0} \sin \frac{\pi x_{1}}{a}, \\ \sigma_{13}^{-} &= \sigma_{13}^{+} = \sigma_{23}^{-} = \sigma_{23}^{+} = \sigma_{33}^{-} = \sigma_{33}^{+} = D_{3}^{-} = \mathbf{0}, \end{aligned}$$
(52)

where $p_0 = 1$ Pa and $\phi_0 = 1$ V. To compare the results derived with the 3D exact solution [16] in the case of mechanical loading, the following dimensionless variables are introduced:

$$\begin{split} \bar{u}_1 &= 100E_T h^2 u_1(a/4,z)/p_0 a^3, \quad \bar{u}_3 = 100E_T h^3 u_3(a/2,z)/p_0 a^4, \\ \bar{\sigma}_{11} &= 10h^2 \sigma_{11}(a/2,z)/p_0 a^2, \quad \bar{\sigma}_{13} = h\sigma_{13}(a/8,z)/p_0 a, \\ \bar{\sigma}_{23} &= 10h\sigma_{23}(7a/8,z)/p_0 a, \quad \bar{\sigma}_{33} = \sigma_{33}(a/2,z)/p_0, \\ \bar{\phi} &= 100E_T d_T h \phi(a/2,z)/p_0 a^2, \quad \bar{D}_3 = h^2 D_3(a/2,z)/d_T p_0 a^2, \end{split}$$
(53)

where a = 1 m, h = 0.2 m, $E_T = 10.3$ GPa and $d_T = 374 \times 10^{-12}$ m/V. In the case of electric loading, we have

$$\begin{split} \tilde{u}_{1} &= 10hu_{1}(a/4,z)/ad_{T}\varphi_{0}, \quad \tilde{u}_{3} &= 10h^{2}u_{3}(a/2,z)/a^{2}d_{T}\varphi_{0}, \\ \tilde{\sigma}_{11} &= h\sigma_{11}(a/2,z)/10E_{T}d_{T}\varphi_{0}, \quad \tilde{\sigma}_{13} &= a\sigma_{13}(a/4,z)/E_{T}d_{T}\varphi_{0}, \\ \tilde{\sigma}_{23} &= a\sigma_{23}(7a/8,z)/E_{T}d_{T}\varphi_{0}, \quad \tilde{\sigma}_{33} &= a^{2}\sigma_{33}(a/2,z)/hE_{T}d_{T}\varphi_{0}, \\ \tilde{\varphi} &= \varphi(a/2,z)/\varphi_{0}, \quad \tilde{D}_{3} &= hD_{3}(a/2,z)/100E_{T}d_{T}^{2}\varphi_{0}. \end{split}$$
(54)

The results from Tables 8–11 show that the SaS method permits the derivation of exact solutions of plane strain electroelasticity for FG piezoelectric angle-ply plates with a prescribed accuracy using the sufficient number of SaS. Figs. 9 and 10 present the distributions of the displacements, transverse stresses and electric displacement through the thickness of a plate for different values of the material index β employing nine SaS inside each layer exactly. As can be seen, the boundary conditions on bottom and top surfaces and the continuity conditions at interfaces for transverse stresses are satisfied with a high accuracy. This statement is confirmed convincingly in Fig. 11 by means of logarithmic errors

$$\begin{aligned} \delta_{\alpha}^{\pm} &= \lg |\bar{\sigma}_{\alpha3}(\pm 0.5)|, \quad \delta_{3}^{-} = \lg |\bar{\sigma}_{33}(-0.5)|, \\ \delta_{3}^{+} &= \lg |\bar{\sigma}_{33}(0.5) - 1|, \quad \delta_{4}^{-} = \lg |\overline{D}_{3}(-0.5)|, \end{aligned}$$
(55)

which characterize the accuracy of fulfilling the boundary conditions for the electric displacement and transverse stresses on outer surfaces for a FG piezoelectric angle-ply plate subjected to mechanical loading.

7. Conclusions

An efficient approach to 3D exact solutions of electroelasticity for FG piezoelectric laminated plates has been proposed. It is based on the new method of SaS located at Chebyshev polynomial nodes inside the plate body. The stress analysis is based on the 3D constitutive equations of electroelasticity and gives the opportunity to obtain the 3D exact solutions for FG piezoelectric laminated thick and thin plates with a specified accuracy.

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