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MECHANICS

The Shear Correction Factor in the Geometrically Nonlinear Theory of Timoshenko Shells

Corresponding Member of the RAS É. I. Grigolyuk* and G. M. Kulikov**

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The necessity of considering transverse shears in the problems of beam bending was first recognized by Timoshenko [1]. On the basis of the virtual work principle, the Timoshenko's theory was extended in [2] to the case of isotropic plates. In order to take into account a nonuniformity in the distribution of transverse shears over cross sections of a shell, the shear correction factor k is introduced in this theory. To date, this factor is usually taken to be equal to $k = \frac{5}{6}$ or $\frac{\pi^2}{12}$ [3]. In [4], it was shown that a value of $k = 1$ is more preferable when the procedure of recovering the transverse components of the stress tensor by integrating the equations of the three-dimensional elasticity theory is used in the linear theory of Timoshenko shells in the approach proposed in [2]. It was this choice that made it possible to construct the mathematically consistent and geometrically noncontradictory linear theory of Timoshenko shells. In this study, the results obtained in [4] are extended to anisotropic shells with finite deflection.

1. Let us consider a thin anisotropic shell of a constant thickness h. We assume that an elastic-symmetry surface parallel to the face surfaces S^- and S^+ exists at each point of the shell. As an initial surface S, we take a shell surface spaced from the face surfaces by δ^- and δ^+; i.e., $h = \delta^+ - \delta^-$. Let the initial surface be associated with the orthogonal curvilinear coordinates α_1 and α_2 measured along the lines of principal curvatures. The coordinate α_3 is measured in the direction of increasing the outer normal to the surface S (see figure).

The equilibrium equations of the elasticity theory for a thin shell, which has a finite deflection and whose face-surface metrics can be identified with the metric of the initial surface, have the form [5]

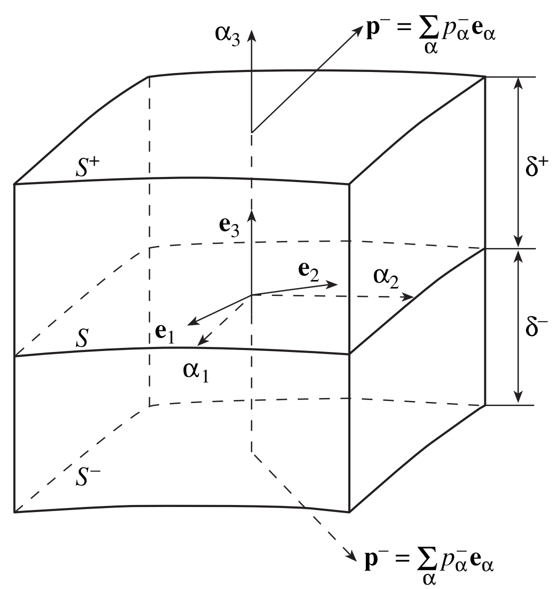
$$\frac{1}{A_i}\frac{\partial\sigma_{ii}}{\partial\alpha_i} + \frac{1}{A_j}\frac{\partial\sigma_{ij}}{\partial\alpha_j} + \frac{\partial\sigma_{i3}}{\partial\alpha_3} + B_i(\sigma_{ii} - \sigma_{jj}) + 2B_j\sigma_{ij} + k_j\Sigma_{i3} = 0 \quad (i \neq j),$$

$$\frac{1}{A_1}\frac{\partial\Sigma_{13}}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial\Sigma_{23}}{\partial\alpha_2} + \frac{\partial\sigma_{33}}{\partial\sigma_3} + B_1\Sigma_{13} + B_2\Sigma_{23} - k_1\sigma_{11} - k_2\sigma_{22} = 0, \quad (1)$$

$$\Sigma_{i3} = \sigma_{i3} + \Theta_1\sigma_{1i} + \Theta_2\sigma_{i2},$$

$$\Theta_i = \frac{1}{A_i}\frac{\partial u_3}{\partial\alpha_i} - k_iu_i, \quad B_i = \frac{1}{A_1A_2}\frac{\partial A_j}{\partial\alpha_j},$$

where u_α are the displacements of the shell points, $\sigma_{\alpha\beta}$ are the stresses, A_i are the Lamé constants, and k_i are the curvatures of the coordinate lines. Hereafter, i, j = 1, 2 and α, β = 1, 2, 3.



Shell element.

* *Moscow State Technical University (MAMI), Moscow, Russia*
** *Tambov State Technical University, Tambov, Russia*

The relationships of the generalized Hooke's law can be written in the form [4]

$$\sigma_{ij} = \sum_{l \le m} b_{ijlm}\varepsilon_{lm}, \quad \sigma_{i3} = \sum_{l} b_{i3l3}\varepsilon_{l3}, \quad i, j, l, m = 1, 2. \tag{2}$$

When constructing our theory, we used the modified Timoshenko hypothesis [4, 6] on the linear distribution of tangential displacements over the shell thickness:

$$u_i = N^-(\alpha_3)v_i^- + N^+(\alpha_3)v_i^+, \quad u_3 = v_3,$$
$$N^-(\alpha_3) = \frac{\delta^+ - \alpha_3}{h}, \quad N^+(\alpha_3) = \frac{\alpha_3 - \delta^-}{h}, \tag{3}$$

where $v_i^{\pm}(\alpha_1, \alpha_2)$ are the tangential displacements of the face surfaces $S^{\pm}$ and $v_3(\alpha_1, \alpha_2)$ is the transverse displacement of the surface S.

Introducing displacements (3) into the stress–strain relationship of the nonlinear theory of elasticity [5] for a thin finite-deflection shell and assuming that the transverse shear strains and tangential strains are distributed, respectively, uniformly and linearly over the shell thickness [7], we obtain

$$\varepsilon_{ij} = N^-(\alpha_3)E_{ij}^- + N^+(\alpha_3)E_{ij}^+, \quad \varepsilon_{i3} = E_{i3}, \quad \varepsilon_{33} = 0,$$
$$E_{ii}^{\pm} = \frac{1}{A_i}\frac{\partial v_j^{\pm}}{\partial \alpha_i} + B_j v_j^{\pm} + k_i v_3 + \frac{1}{2}(\theta_i^{\pm})^2 \quad (i \ne j),$$
$$E_{12}^{\pm} = \frac{1}{A_1}\frac{\partial v_2^{\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial v_1^{\pm}}{\partial \alpha_2} - B_2 v_1^{\pm} - B_1 v_2^{\pm} + \theta_1^{\pm}\theta_2^{\pm}, \tag{4}$$
$$E_{i3} = \beta_i - \theta_i,$$
$$\beta_i = \frac{1}{h}(v_i^+ - v_i^-), \quad \theta_i^{\pm} = k_i v_1^{\pm} - \frac{1}{A_i}\frac{\partial v_3}{\partial \alpha_i},$$
$$\theta_i = \frac{1}{2}(\theta_i^- + \theta_i^+).$$

Multiplying the first two of the equilibrium Eqs. (1) by the functions of the shape $N^{\pm}(\alpha_3)$ and integrating the resulting equations along with the third equation with respect to the transverse coordinate from δ^- to δ^+ with allowance for the boundary conditions at the face surfaces $S^{\pm}$,

$$\sigma_{\alpha 3}(\delta^{\pm}) = p_{\alpha}^{\pm}, \tag{5}$$

we obtain the expressions consistent with the virtual work principle. Here, $p_{\alpha}^{\pm}$ are the components of the surface-load vectors $\mathbf{p}^{\pm}$ at the face surfaces $S^{\pm}$ (see figure). As a result, taking into account Eqs. (3) and (4), we arrive at the following nonlinear equilibrium equations for the shell with respect to the stress and moment resultants:

$$\frac{1}{A_i}\frac{\partial H_{ii}^{\pm}}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial H_{ij}^{\pm}}{\partial \alpha_j} + B_i(H_{ii}^{\pm} - H_{jj}^{\pm}) + 2B_j H_{ij}^{\pm}$$
$$+ k_i S_{i3}^{\pm} \mp \frac{1}{h}T_{i3} = \mp p_i^{\pm} \quad (i \ne j),$$
$$\frac{1}{A_1}\frac{\partial N_{13}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial N_{23}}{\partial \alpha_2} + B_1 N_{13} + B_2 N_{23} \tag{6}$$
$$- k_1 T_{11} - k_2 T_{22} = p_3^- - p_3^+,$$

where $T_{i\alpha}$, $H_{ij}^{\pm}$, H_{ij}^{pq}, N_{i3}, and $S_{i3}^{\pm}$ are the stress and generalized moment resultants determined by the formulas

$$N_{i3} = T_{i3} - \theta_1^- H_{1i}^- - \theta_1^+ H_{1i}^+ - \theta_2^- H_{i2}^- - \theta_2^+ H_{i2}^+,$$
$$S_{i3}^- = \frac{1}{2}T_{i3} - \theta_1^- H_{1i}^{00} - \theta_1^+ H_{1i}^{01} - \theta_2^- H_{i2}^{00} - \theta_2^+ H_{i2}^{01},$$
$$S_{i3}^+ = \frac{1}{2}T_{i3} - \theta_1^- H_{1i}^{01} - \theta_1^+ H_{1i}^{11} - \theta_2^- H_{i2}^{01} - \theta_2^+ H_{i2}^{11},$$
$$T_{i\alpha} = \int_{\delta^-}^{\delta^+} \sigma_{i\alpha} d\alpha_3, \quad H_{ij}^{\pm} = \int_{\delta^-}^{\delta^+} \sigma_{ij} N^{\pm}(\alpha_3) d\alpha_3, \tag{7}$$
$$H_{ij}^{pq} = \int_{\delta^-}^{\delta^+} \sigma_{ij}[N^-(\alpha_3)]^{2-p-q}[N^+(\alpha_3)]^{p+q} d\alpha_3,$$
$$p, q = 0, 1.$$

The elasticity relationships for the stress and generalized moment resultants (7) with allowance for Eqs. (2) and (4) can be represented in the form

$$H_{ij}^{00} = \frac{1}{12}h\sum_{l \le m} b_{ijlm}(3E_{lm}^- + E_{lm}^+),$$
$$H_{ij}^{01} = \frac{1}{12}h\sum_{l \le m} b_{ijlm}(E_{lm}^- + E_{lm}^+),$$
$$H_{ij}^{11} = \frac{1}{12}h\sum_{l \le m} b_{ijlm}(E_{lm}^- + 3E_{lm}^+), \tag{8}$$
$$H_{ij}^- = H_{ij}^{00} + H_{ij}^{01}, \quad H_{ij}^+ = H_{ij}^{01} + H_{ij}^{11},$$
$$T_{ij} = H_{ij}^- + H_{ij}^+, \quad T_{i3} = h\sum_{l} k_{il} b_{i3l3} e_{l3},$$

where we take $k_{il} = 1$ for the shear correlation factors. Formula (8) for the transverse forces T_{i3} has simple meaning: the elasticity relationships for transverse shear stresses (2) in the theory of Timoshenko shells are satisfied not pointwise but integrally over the shell thickness [6–8].

Integrating elasticity-theory Eqs. (1) with respect to the transverse coordinate from δ^- to α_3 with allowance for boundary conditions (5) at the face surface S^-, we arrive at the following formulas for the transverse components of the stress tensor:

$$\sigma_{i3} = p_i^- - \frac{1}{A_i}\frac{\partial Q_{ii}}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial Q_{ij}}{\partial \alpha_j} - B_i(Q_{ii} - Q_{jj}) - 2B_j Q_{ij} - k_i R_{i3} \quad (i \neq j),$$
$$\sigma_{33} = p_3^- - \frac{1}{A_1}\frac{\partial R_{13}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial R_{23}}{\partial \alpha_2} - B_1 R_{13} - B_2 R_{23} + k_1 Q_{11} + k_2 Q_{22}, \tag{9}$$

where

$$R_{i3} = Q_{i3} - \theta_1^- M_{1i}^- - \theta_1^+ M_{1i}^+ - \theta_2^- M_{i2}^- - \theta_2^+ M_{i2}^+,$$
$$Q_{i\alpha} = \int_{\delta^-}^{\alpha_3} \sigma_{i\alpha} d\alpha_3, \quad M_{ij}^{\pm} = \int_{\delta^-}^{\alpha_3} \sigma_{ij} N^{\pm}(\alpha_3) d\alpha_3. \tag{10}$$

In view of the equalities $Q_{i\alpha}(\delta^+) = T_{i\alpha}$, $M_{ij}^{\pm}(\delta^+) = H_{ij}^{\pm}$ and shell equilibrium equations (6), the boundary conditions (5) at the face surface S^+ result immediately from Eqs. (9).

2. Now, we discuss the question (that is of importance for the theory of Timoshenko shells) of whether or not the stress field, which is specified by Eqs. (2) and (9) and was found by solving the problem, satisfies equilibrium shell equations (6). The shell equilibrium equations (6) can be no longer exactly satisfied with the transverse forces T_{i3}^s evaluated on the basis of the refined transverse shear stress σ_{i3} determined by Eq. (9). The reason is that the transverse forces T_{i3} evaluated on the basis of Hooke's law (2), i.e., by Eq. (8), can generally differ from T_{i3}^s.

To solve this problem, we use the following formulas obtained from Eqs. (2), (4), (8), and (10):

$$\int_{\delta^-}^{\delta^+} Q_{ij} d\alpha_3 = hH_{ij}^-, \quad \int_{\delta^-}^{\delta^+} Q_{i3} d\alpha_3 = \frac{1}{2} hT_{i3},$$
$$\int_{\delta^-}^{\delta^+} M_{ij}^- d\alpha_3 = hH_{ij}^{00}, \quad \int_{\delta^-}^{\delta^+} M_{ij}^+ d\alpha_3 = hH_{ij}^{01}.$$

With allowance for these formulas and Eqs. (9), we obtain

$$T_{i3}^s = \int_{\delta^-}^{\delta^+} \sigma_{i3} d\alpha_3 = h\Big[p_i^- - \frac{1}{A_i}\frac{\partial H_{ii}^-}{\partial \alpha_j} - \frac{1}{A_j}\frac{\partial H_{ij}^-}{\partial \alpha_j} - B_i(H_{ii}^- - H_{jj}^-) - 2B_j H_{ij}^- - k_i S_{i3}^-\Big] \quad (i \neq j). \tag{11}$$

Taking the shell equilibrium equations (6) into account, we derive from Eq. (11) that $T_{i3}^s = T_{i3}$, as we wished prove.

Thus, on the basis of the physically clear supposition that the equations of Hooke's law (2) for transverse shear stresses are integrally satisfied, we succeeded in constructing the self-consistent geometrically nonlinear theory of Timoshenko shells. In this theory, the nonlinear shell equilibrium equations (6) and Eqs. (9) are satisfied together. For this reason, $k_{il} = 1$ was taken in Eq. (8) for the transverse forces. In conclusion, it must be emphasized that, from the standpoint of the approach developed in this study, efforts on constructing a geometrically nonlinear theory of Timoshenko shells on the basis of various ways of evaluating the shear correlation factors k_{il} will lead to a mathematically inconsistent and contradictory theory.

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