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MECHANICS

Solution of Static Problems for a Three-Dimensional Elastic Shell

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As is known, the conventional way of constructing the theory of shells consists in the expansion of displacements into power series with respect to the transverse coordinate θ_3 counted along the external normal to the shell middle surface. For the approximate representation of the field of displacements, it is possible to use finite segments of power series because the principal purpose of the theory of shells consists in obtaining approximate solutions of the problems of the three-dimensional theory of elasticity. The idea of this approach goes back to Cauchy [1]. This way is used for specifying the theory of shells taking into account transverse shears [2–4]. The theory of shells was also developed on the basis of the expansion of the field of displacements into series in the Legendre polynomials with respect to the coordinate θ_3 [5]. However, the apparent advantage of this theory is lost in its application to the problems of statics of thick shells in which it is necessary to retain a sufficiently large number of terms in the corresponding expansions for obtaining comprehensible results. An alternative approach is associated with consideration of the reference surfaces $\Omega^1, \Omega^2, \ldots, \Omega^N$ parallel to the middle surface in the shell bulk with the purpose of using the displacement vectors $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N$ of these surfaces as the desired functions. The development of this approach for the case of three equidistance reference surfaces is given in [6, 7]. Here we investigated the general case with an arbitrary arrangement of the reference surfaces in the shell.

1. We consider the shell of a constant thickness h. The middle surface Ω is related to the curvilinear orthogonal coordinates θ_1 and θ_2 counted along the principal-curvature lines. We present the basic vectors of the reference surfaces Ω^I as

$$\mathbf{g}_\alpha^I = \mathbf{R}_{,\alpha}^I = A_\alpha c_\alpha^I \mathbf{e}_\alpha, \quad \mathbf{g}_3^I = \mathbf{e}_3, \quad c_\alpha^I = 1 + k_\alpha \theta_3^I, \tag{1}$$

where $\mathbf{R}^I$ are the radius vectors of the surfaces Ω^I (Fig. 1), $\mathbf{e}_\alpha$ are the unit vectors tangential to the coordinate lines θ_α; $\mathbf{e}_3$ is the unit vector of the external normal to the middle surface, A_α are the Lamé parameters, k_α are the principal curvatures, and θ_3^I are the transverse coordinates of the surfaces Ω^I; in this case, $\theta_3^I = -h/2$ and $\theta_3^N = h/2$. From now on, the subscripts $\alpha, \beta = 1, 2$; $i, j, k, m = 1, 2, 3$; $I, J, K = 1, 2, \ldots, N$.

We find the basic vectors of the shell reference surfaces in a deformed state from the formulas

$$\bar{\mathbf{g}}_\alpha^I = \bar{\mathbf{R}}_{,\alpha}^I = \mathbf{g}_\alpha^I + \mathbf{u}_{,\alpha}^I, \quad \bar{\mathbf{g}}_3^I = \mathbf{e}_3 + \boldsymbol{\beta}^I, \tag{2}$$

$$\mathbf{u}^I = \mathbf{u}(\theta_3^I), \quad \boldsymbol{\beta}^I = \mathbf{u}_{,3}(\theta_3^I), \tag{3}$$

where $\bar{\mathbf{R}}^I$ are the radius vectors of the surfaces Ω^I in the deformed state (Fig. 1), $\mathbf{u}^I(\theta_1, \theta_2)$ are the vectors of displacements of the surfaces Ω^I, and $\boldsymbol{\beta}^I(\theta_1, \theta_2)$ are the values of the derivative of the vector of displacements along the coordinate θ_3 on the surfaces Ω^I.

The components of the deformation tensor of on the shell reference surfaces have the form

$$2\varepsilon_{ij}^I = 2\varepsilon_{ij}(\theta_3^I) = \frac{1}{A_i A_j c_i^I c_j^I}(\bar{\mathbf{g}}_i^I \cdot \bar{\mathbf{g}}_j^I - \mathbf{g}_i^I \cdot \mathbf{g}_j^I), \tag{4}$$

where $A_3 = 1$ and $c_3^I = 1$. Introducing basic vectors (1), (2), in deformation relations (4) of the spatial theory of elasticity and neglecting nonlinear terms, we obtain

$$\begin{aligned} 2\varepsilon_{\alpha\beta}^I &= \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_\beta + \frac{1}{A_\beta c_\beta^I}\mathbf{u}_{,\beta}^I \cdot \mathbf{e}_\alpha, \\ 2\varepsilon_{\alpha 3}^I &= \boldsymbol{\beta}^I \cdot \mathbf{e}_\alpha + \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_3, \quad \varepsilon_{33}^I = \boldsymbol{\beta}^I \cdot \mathbf{e}_3. \end{aligned} \tag{5}$$

Let us present the vectors $\mathbf{u}^I$ and $\boldsymbol{\beta}^I$ in the orthonormalized basis $\mathbf{e}_i$:

$$\mathbf{u}^I = \sum_i u_i^I \mathbf{e}_i, \quad \boldsymbol{\beta}^I = \sum_i \beta_i^I \mathbf{e}_i. \tag{6}$$

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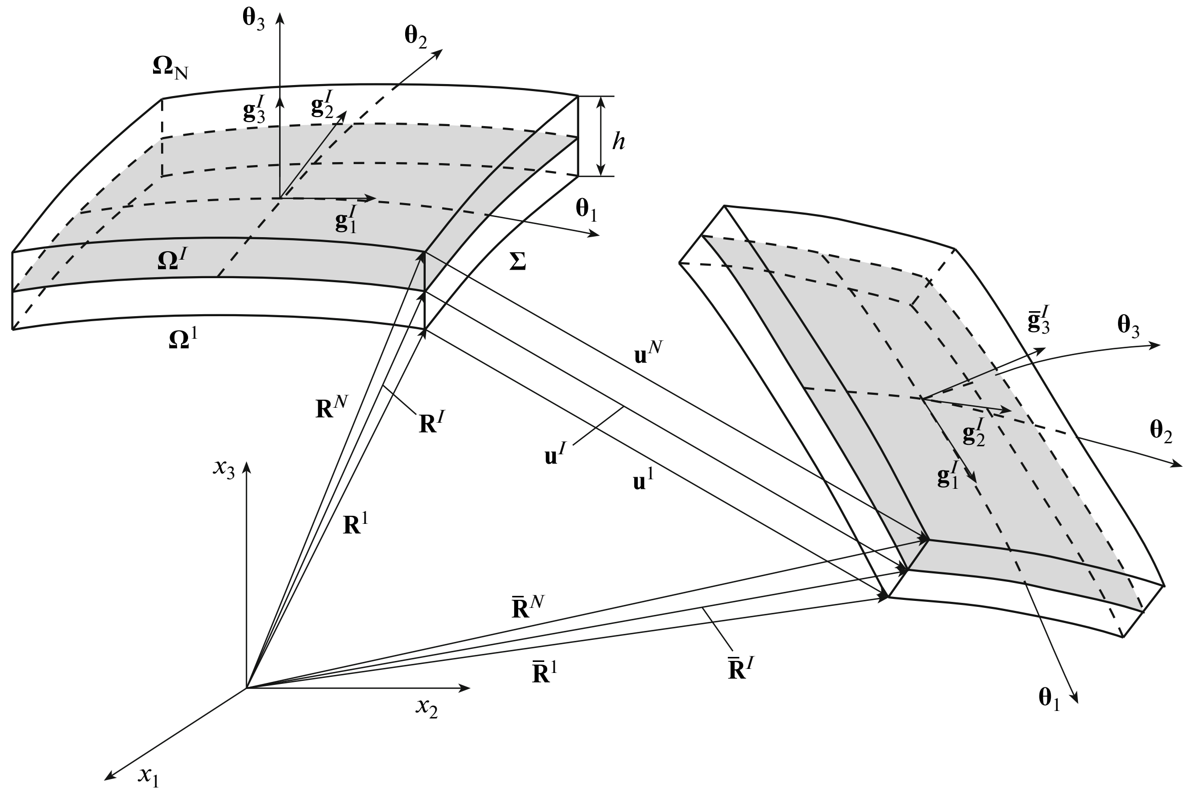

Fig. 1. Initial and deformed shell configurations.

From the first expansion in Eqs. (6) with taking into account the known formulas of differentiation of basic vectors $\mathbf{e}_i$ with respect to the coordinates θ_α [1], it follows that

$$\frac{1}{A_\alpha}\mathbf{u}^I_{,\alpha} = \sum_i \lambda^I_{i\alpha}\mathbf{e}_i, \tag{7}$$

where

$$\lambda^I_{\alpha\alpha} = \frac{1}{A_\alpha}u^I_{\alpha,\alpha} + B_\alpha u^I_\beta + k_\alpha u^I_3, \quad \lambda^I_{\beta\alpha} = \frac{1}{A_\alpha}u^I_{\beta,\alpha} - B_\alpha u^I_\alpha,$$
$$\lambda^I_{3\alpha} = \frac{1}{A_\alpha}u^I_{3,\alpha} - k_\alpha u^I_\alpha, \quad B_\alpha = \frac{1}{A_\alpha A_\beta}A_{\alpha,\beta} \ (\beta \neq \alpha). \tag{8}$$

Substituting Eqs. (6) and (7) into Eqs. (5), we come to the scalar form of the deformation relations

$$2\varepsilon^I_{\alpha\beta} = \frac{1}{c^I_\beta}\lambda^I_{\alpha\beta} + \frac{1}{c^I_\alpha}\lambda^I_{\beta\alpha}, \quad 2\varepsilon^I_{\alpha3} = \beta^I_\alpha + \frac{1}{c^I_\alpha}\lambda^I_{3\alpha},$$
$$\varepsilon^I_{33} = \beta^I_3. \tag{9}$$

2. We note that, up to this moment, we have made no assumptions about the character of distribution of the displacement and deformation fields in the shell. Let the displacements be distributed in the shell transverse direction according to the following law:

$$\mathbf{u} = \sum_I L^I \mathbf{u}^I, \tag{10}$$

where $L^I(\theta_3)$ are the Lagrange polynomials of the $(N-1)$th degree determined from the known formula

$$L^I = \prod_{J \neq I} \frac{\theta_3 - \theta^J_3}{\theta^I_3 - \theta^J_3}, \tag{11}$$

with $L^I(\theta^J_3) = 1$ for $J = I$, and $L^I(\theta^J_3) = 0$ for $J \neq I$.

From Eqs. (3), (6), and (10), we have

$$\beta^I_i = \sum_J M^J(\theta^I_3)u^J_i, \tag{12}$$

where $M^J = L^J_{,3}$ are the polynomials of the $(N-2)$th degree; their values on the reference surfaces Ω^I can be found according to Eq. (11) from the formulas

$$M^J\left(\theta^I_3\right) = \frac{1}{\theta^J_3 - \theta^I_3}\prod_{K \neq I,J}\frac{\theta^I_3 - \theta^K_3}{\theta^J_3 - \theta^K_3} \quad (J \neq I),$$
$$M^I\left(\theta^I_3\right) = -\sum_{J \neq I} M^J\left(\theta^I_3\right). \tag{13}$$

Thus, the defining functions β^I_i of the proposed theory of shells are presented as the linear combination of displacements of the reference surfaces u^I_i.

The following step consists in the choice of the law of distribution of deformations over the shell thickness. It is obvious that the distribution of deformations

Table 1. Results of calculation for a thick shell ($R/h = 2$)

N	$U_3(0)$	$S_{22}(0.5)$	$-S_{22}(-0.5)$	$S_{23}(0)$
3	0.889	1.125	0.701	0.448
5	0.977	1.839	2.056	0.577
7	0.995	1.903	2.399	0.547
9	0.998	1.913	2.455	0.555
Solution [9]	0.999	1.907	2.455	0.555

in the transverse direction should be correlated with displacement distribution (10), i.e.,

$$\varepsilon_{ij} = \sum_I L^I \varepsilon_{ij}^I. \tag{14}$$

3. We substitute deformations (14) in the principle of virtual work and, introducing the resulting stresses

$$H_{ij}^I = \int_{-h/2}^{h/2} \sigma_{ij} L^I c_1 c_2 d\theta_3, \quad c_\alpha = 1 + k_\alpha \theta_3, \tag{15}$$

we come to the variational equation

$$\iint_\Omega \left[\sum_I \sum_{i,j} H_{ij}^I \delta\varepsilon_{ij}^I - \sum_i \left(c_1^N c_2^N p_i^+ \delta u_i^N - c_1^1 c_2^1 p_i^- \delta u_i^1 \right) \right] \times A_1 A_2 d\theta_1 d\theta_2 = \delta W_\Sigma, \tag{16}$$

where p_i^- and p_i^+ are the surface loads acting on the internal and external shell surfaces; W_Σ is the work of external forces acting on the lateral surface Σ.

We restrict ourselves to consideration of linearly elastic materials for which the relations of the generalized Hooke law are applicable:

$$\sigma_{ij} = \sum_{k,m} C_{ijkm} \varepsilon_{km}. \tag{17}$$

Further, we introduce stresses (17) in Eq. (15) and, taking into account the designation

$$D_{ijkm}^{IJ} = C_{ijkm} \int_{-h/2}^{h/2} L^I L^J c_1 c_2 d\theta_3, \tag{18}$$

obtain the expression for calculating the resulting stresses

$$H_{ij}^I = \sum_J \sum_{k,m} D_{ijkm}^{IJ} \varepsilon_{km}^J. \tag{19}$$

Remark. Definite integral (18) can be calculated exactly by using the Gauss quadrature formulas with the $(2N + 1)$th order of accuracy because the integrated function is the polynomial of the $2N$th degree.

4. Variational Eq. (16) taking into account Eqs. (18), (19) is the basis for constructing the geometrically exact bilinear finite element of the shell [7, 8]. The term "geometrically exact" means that the shell middle surface is described by analytically set functions, in particular, splines; i.e., the surface parametrization is considered known. In this case, the analytical integration was used within the limits of a finite element, which is the prerogative of the geometrically exact shell element [8].

As an example, we consider the cylindrical bend of a hinge-supported cylindrical shell under the action of a transverse load $p_3^+ = p_0 \sin 3\theta_2$, where θ_2 is the peripheral coordinate varying from 0 to $\pi/3$. To satisfy the boundary conditions at the end faces of the shell, it is accepted that $u_3^I = 0$, which is equivalent to $u_3 = 0$ according to Eq. (10). The cylindrical shell of radius $R = 10$ is fabricated from a composite reinforced in the peripheral direction with the following mechanical parameters [9]: $E_L = 25E_T$, $G_{LT} = 0.5E_T$, $G_{TT} = 0.2E_T$, $E_T = 10^6$, and $\nu_{LT} = \nu_{TT} = 0.25$. The subscripts L and T correspond to the reinforcement and transverse directions, respectively. For comparison with the analytical solution of the plane problem of the theory of elasticity [9], we introduce the dimensionless sizes

$$U_3 = \frac{10E_T h^3 u_3(\pi/6, z)}{p_0 R^4}, \quad S_{22} = \frac{h^2 \sigma_{22}(\pi/6, z)}{p_0 R^2},$$
$$S_{23} = \frac{h\sigma_{23}(0, z)}{p_0 R}, \quad S_{33} = \frac{\sigma_{33}(\pi/6, z)}{p_0}, \quad z = \frac{\theta_3}{h}.$$

Because of the symmetry of the problem, we considered a half of the shell, which was simulated with the help of 64 finite elements describing the plane deformed state of the shell. The data in Tables 1 and 2 shows that, for an appropriate choice of equidistance

Table 2. Results of calculation of a cylindrical shell

$\frac{R}{h}$	$N = 7$				Exact solution [9]		
	$U_3(0)$	$S_{22}(0.5)$	$S_{23}(0)$	$S_{23}(-0.5)$	$U_3(0)$	$S_{22}(0.5)$	$S_{23}(0)$
4	0.312	1.079	0.571	0.028	0.312	1.079	0.572
10	0.115	0.806	0.579	0.006	0.115	0.807	0.579
50	0.077	0.752	0.568	0.001	0.077	0.752	0.568
100	0.076	0.750	0.565	0.000	0.076	0.751	0.565

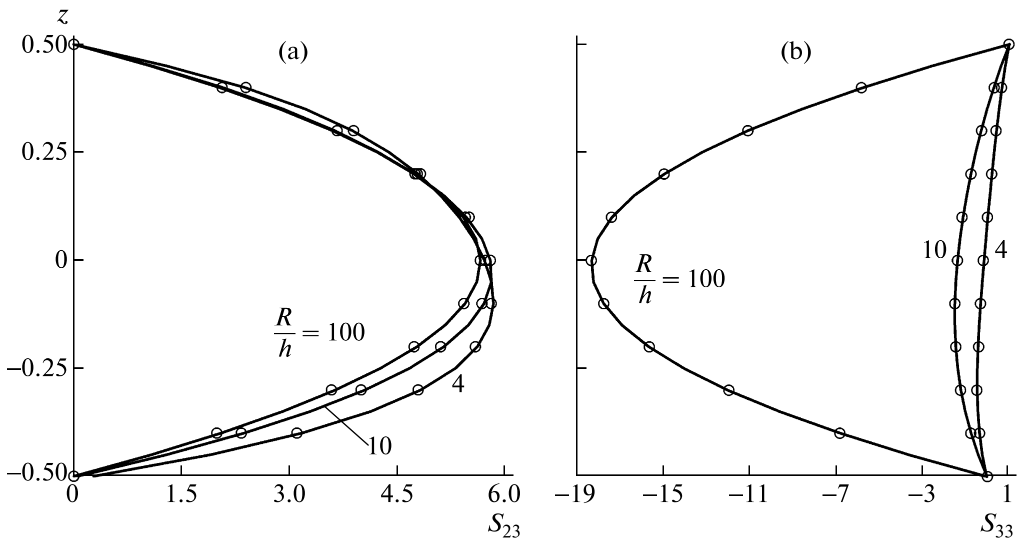

Fig. 2. Distribution of transverse stresses (a) S_{23} and (b) S_{33} over the shell thickness: (points) the exact solution [9] and (curve) this theory of shells for $N = 7$.

reference surfaces, we can achieve good agreement with the analytical solution even in the case of a thick shell. The transverse-stress distribution over the shell thickness for the choice of seven equidistance reference surfaces also indicates the high potential of the proposed theory because the boundary conditions on the shell face surfaces are satisfied with an accuracy sufficient for practice (see Fig. 2, Table 2).

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