Geometrically exact four-node piezoelectric solid-shell element

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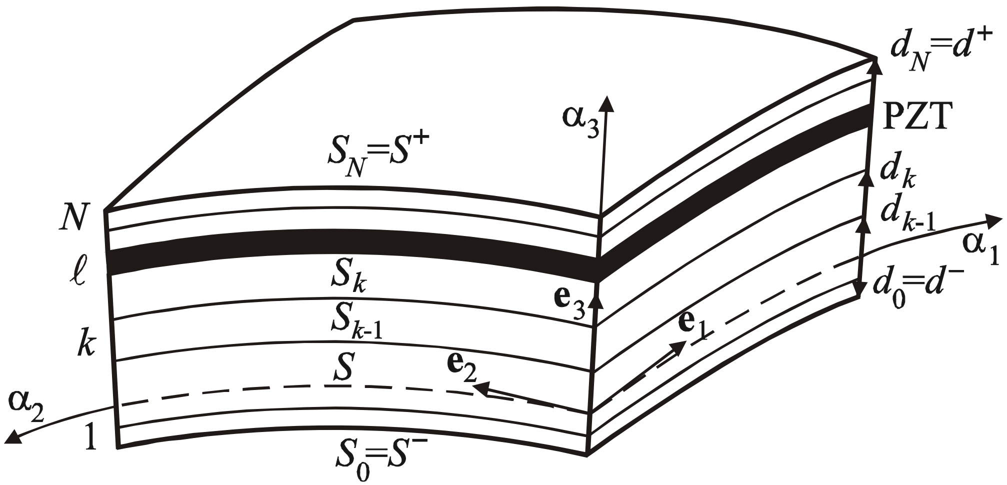

Fig. 1. Piezoelectric laminated shell

Timoshenko-Mindlin kinematics (Kulikov, IJSS, 2001):

$$\mathbf{u} = N^-\mathbf{u}^- + N^+\mathbf{u}^+ \tag{1}$$

$$\mathbf{u}^\pm = \sum_i u_i^\pm \mathbf{e}_i \tag{2}$$

$$N^- = \frac{1}{h}\left(d^+ - \alpha_3\right) \tag{3}$$

$$N^+ = \frac{1}{h}\left(\alpha_3 - d^-\right)$$

$\mathbf{u}^\pm(\alpha_1, \alpha_2)$ – displacement vectors of face surfaces $S^\pm$; h – shell thickness; $i = 1, 2, 3$

Strain-displacement relationships (Kulikov and Plotnikova, MAMS, 2005):

$$\varepsilon_{\alpha\beta} = N^{-}\varepsilon^{-}_{\alpha\beta} + N^{+}\varepsilon^{+}_{\alpha\beta}, \qquad \varepsilon_{\alpha 3} = N^{-}\varepsilon^{-}_{\alpha 3} + N^{+}\varepsilon^{+}_{\alpha 3}, \qquad \varepsilon_{33} = \bar{\varepsilon}_{33} \tag{4}$$

$\varepsilon^{\pm}_{\alpha\beta}$, $\varepsilon^{\pm}_{\alpha 3}$ – in-plane and transverse shear strains of face surfaces $S^{\pm}$:

$$2\varepsilon^{\pm}_{\alpha\beta} = \zeta^{\pm}_{\alpha}\lambda^{\pm}_{\alpha\beta} + \zeta^{\pm}_{\beta}\lambda^{\pm}_{\beta\alpha}, \quad 2\varepsilon^{\pm}_{\alpha 3} = \zeta^{\pm}_{\alpha}\beta_{\alpha} - \theta^{\pm}_{\alpha}, \qquad \bar{\varepsilon}_{33} = \beta_3 \tag{5}$$

$$\lambda^{\pm}_{\alpha\alpha} = \left(\frac{1}{A_\alpha}u^{\pm}_{\alpha}\right)_{,\alpha} + B_{\alpha\alpha}u^{\pm}_{\alpha} + B_{\alpha\beta}u^{\pm}_{\beta} + k_\alpha u^{\pm}_{3}, \qquad \lambda^{\pm}_{\beta\alpha} = \left(\frac{1}{A_\alpha}u^{\pm}_{\beta}\right)_{,\alpha} + B_{\alpha\alpha}u^{\pm}_{\beta} - B_{\alpha\beta}u^{\pm}_{\alpha} \qquad (\beta \neq \alpha) \tag{6}$$

$$\theta^{\pm}_{\alpha} = -\left(\frac{1}{A_\alpha}u^{\pm}_{3}\right)_{,\alpha} - B_{\alpha\alpha}u^{\pm}_{3} + k_\alpha u^{\pm}_{\alpha}, \quad \beta_i = \frac{1}{h}\left(u^{+}_{i} - u^{-}_{i}\right), \qquad B_{\alpha\beta} = \frac{1}{A_\alpha A_\beta}A_{\alpha,\beta}, \qquad \varsigma^{\pm}_{\alpha} = 1 + k_\alpha d^{\pm}$$

A_α, k_α – Lamé coefficients and principal curvatures of the reference surface S; $\alpha, \beta = 1, 2$

Assumed displacement-independent strains:

$$\varepsilon^{AS}_{\alpha\beta} = N^{-}\hat{\varepsilon}^{-}_{\alpha\beta} + N^{+}\hat{\varepsilon}^{+}_{\alpha\beta}, \qquad \varepsilon^{AS}_{\alpha 3} = N^{-}\hat{\varepsilon}^{-}_{\alpha 3} + N^{+}\hat{\varepsilon}^{+}_{\alpha 3}, \qquad \varepsilon^{AS}_{33} = \hat{\varepsilon}_{33} \tag{7}$$

Electric potential inside piezoelectric layer:

$$\varphi_\ell = N_\ell^- \varphi_\ell^- + N_\ell^+ \varphi_\ell^+ \qquad (8)$$

$$N_\ell^- = \frac{1}{h_\ell}(d_\ell - \alpha_3), \qquad N_\ell^+ = \frac{1}{h_\ell}(\alpha_3 - d_{\ell-1})$$

$\varphi_\ell^\pm(\alpha_1, \alpha_2)$ – electric potentials on face surfaces $S_\ell^\pm$ of the PZT layer

$h_\ell = d_\ell - d_{\ell-1}$ – layer thickness; $\ell \in \{1, 2, \ldots, N\}$

Electric field:

$$\mathbf{E}^{(\ell)} = -\nabla \varphi_\ell \qquad (9)$$

$$E_\alpha^{(\ell)} = -N_\ell^- \varphi_{\ell,\alpha}^- - N_\ell^+ \varphi_{\ell,\alpha}^+, \qquad E_3^{(\ell)} = -\frac{1}{h_\ell}\left(\varphi_\ell^+ - \varphi_\ell^-\right)$$

Constitutive equations:

$$\boldsymbol{\varepsilon} = \mathbf{A}\boldsymbol{\sigma} + \mathbf{d}^{\mathrm{T}}\mathbf{E} \quad (10)$$

$$\mathbf{D} = \mathbf{d}\boldsymbol{\sigma} + \boldsymbol{\varsigma}\mathbf{E} \quad (11)$$

$$\boldsymbol{\sigma} = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{13}\ \sigma_{12}]^{\mathrm{T}}, \qquad \boldsymbol{\varepsilon} = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ 2\varepsilon_{23}\ 2\varepsilon_{13}\ 2\varepsilon_{12}]^{\mathrm{T}} \quad (12)$$

$$\mathbf{D} = [D_1\ D_2\ D_3]^{\mathrm{T}}, \qquad \mathbf{E} = [E_1\ E_2\ E_3]^{\mathrm{T}}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \underline{A_{13}} & 0 & 0 & A_{16} \\ & A_{22} & \underline{A_{23}} & 0 & 0 & A_{26} \\ & & A_{33} & 0 & 0 & \underline{A_{36}} \\ & & & A_{44} & A_{45} & 0 \\ & & & & A_{55} & 0 \\ \text{sym.} & & & & & A_{66} \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} 0 & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & d_{25} & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & d_{36} \end{bmatrix}, \qquad \boldsymbol{\varsigma} = \begin{bmatrix} \varsigma_{11} & \varsigma_{12} & 0 \\ & \varsigma_{22} & 0 \\ \text{sym.} & & \varsigma_{33} \end{bmatrix}$$

$\boldsymbol{\sigma}$ – stress vector; $\boldsymbol{\varepsilon}$ – strain vector; **D** – electric displacement vector

A – elastic compliance matrix; **d** – piezoelectric matrix; $\boldsymbol{\varsigma}$– dielectric matrix

Modified constitutive equations (Lee et al., 2003):

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} - \mathbf{e}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{C} = \tilde{\mathbf{A}}^{-1}, \quad \mathbf{e} = \mathbf{d}\mathbf{C} \tag{13}$$

$$\mathbf{D} = \mathbf{e}\boldsymbol{\varepsilon} + \in \mathbf{E}, \quad \in = \varsigma - \mathbf{d}\mathbf{C}\mathbf{d}^{\mathrm{T}} \tag{14}$$

$\tilde{\mathbf{A}}$ – modified compliance matrix whose components $A_{i3} = 0$

Mixed variational principle:

$$\delta\Pi_{\mathrm{HW}} = 0 \tag{15}$$

3D Hu-Washizu functional:

$$\Pi_{\mathrm{HW}} = \iiint_V \left[\frac{1}{2}\left(\boldsymbol{\varepsilon}^{\mathrm{AS}}\right)^{\mathrm{T}} \mathbf{C}\boldsymbol{\varepsilon}^{\mathrm{AS}} + \left(\boldsymbol{\varepsilon}^{\mathrm{AS}}\right)^{\mathrm{T}} \mathbf{e}^{\mathrm{T}}\nabla\varphi - \frac{1}{2}(\nabla\varphi)^{\mathrm{T}} \in \nabla\varphi - \boldsymbol{\sigma}^{\mathrm{T}}\left(\boldsymbol{\varepsilon}^{\mathrm{AS}} - \boldsymbol{\varepsilon}\right) - \mathbf{b}^{\mathrm{T}}\mathbf{u}\right]\mathrm{d}V - W \tag{16}$$

$$\mathbf{u} = [u_1 \ u_2 \ u_3]^{\mathrm{T}}, \ \nabla\varphi = [\varphi_{,1} \ \varphi_{,2} \ \varphi_{,3}]^{\mathrm{T}}, \ \mathbf{b} = [b_1 \ b_2 \ b_3]^{\mathrm{T}}, \ \boldsymbol{\varepsilon}^{\mathrm{AS}} = \left[\varepsilon_{11}^{\mathrm{AS}} \ \varepsilon_{22}^{\mathrm{AS}} \ \varepsilon_{33}^{\mathrm{AS}} \ 2\varepsilon_{23}^{\mathrm{AS}} \ 2\varepsilon_{13}^{\mathrm{AS}} \ 2\varepsilon_{12}^{\mathrm{AS}}\right]^{\mathrm{T}}$$

b – body force vector; *W* – external work

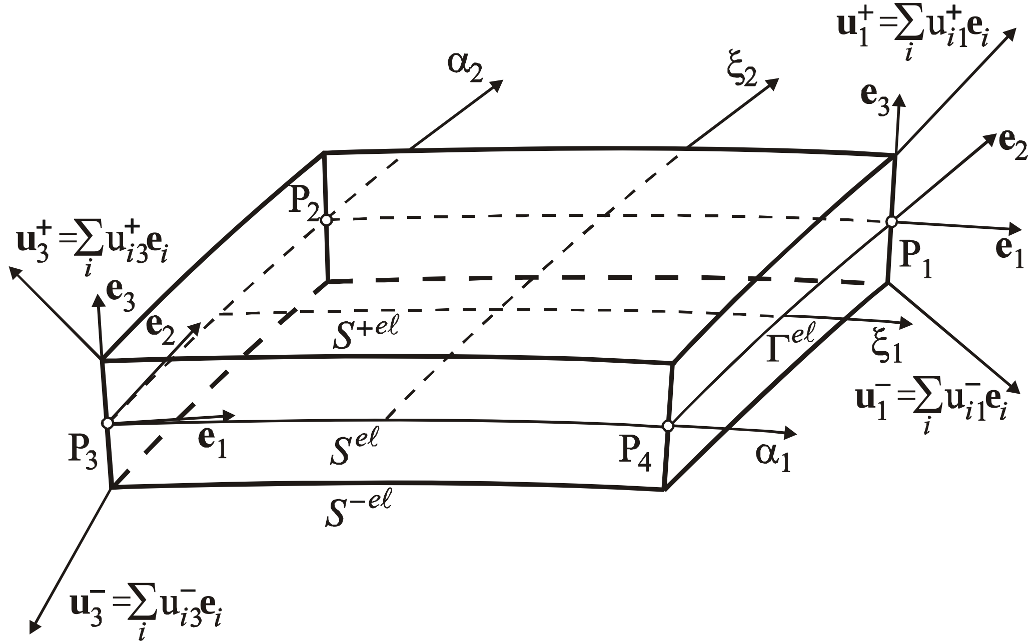

Fig. 2. Geometrically exact four-node piezoelectric solid-shell element

2D Hu-Washizu functional:

$$\Pi_{HW}^{e\ell} = \int_{-1}^{1}\int_{-1}^{1}\Big[\frac{1}{2}\hat{\boldsymbol{\varepsilon}}^{T}\mathbf{D}_{M}\hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\varepsilon}}^{T}\mathbf{D}_{ME}^{(\ell)}\boldsymbol{\Psi}_{\ell} + \frac{1}{2}\boldsymbol{\Psi}_{\ell}^{T}\mathbf{D}_{E}^{(\ell)}\boldsymbol{\Psi}_{\ell} - \mathbf{H}^{T}\left(\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}\right) - \mathbf{v}^{T}\left(\mathbf{P}_{B} + \mathbf{P}_{S}\right) - \mathbf{Q}_{S}^{(\ell)}\boldsymbol{\varphi}_{\ell}\Big]\,\overline{A}_1^{e\ell}\,\overline{A}_2^{e\ell}\,d\xi_1 d\xi_2 \quad (17)$$

$\mathbf{D}_M$, $\mathbf{D}_{ME}^{(\ell)}$, $\mathbf{D}_E^{(\ell)}$ – mechanical, piezoelectric and dielectric constitutive matrices

$\mathbf{H}$ – stress resultant vector; $\mathbf{P}_B$ – body force vector; $\mathbf{P}_S$ – surface traction vector

$$\mathbf{v} = \left[u_1^- \; u_1^+ \; u_2^- \; u_2^+ \; u_3^- \; u_3^+\right]^{\mathrm{T}}, \qquad \varphi_\ell = \left[\varphi_\ell^- \; \varphi_\ell^+\right]^{\mathrm{T}}, \qquad \boldsymbol{\Psi}_\ell = \left[\varphi_\ell^{\mathrm{T}} \; \varphi_{\ell,1}^{\mathrm{T}} \; \varphi_{\ell,2}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{18}$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{11}^- \; \varepsilon_{11}^+ \; \varepsilon_{22}^- \; \varepsilon_{22}^+ \; 2\varepsilon_{12}^- \; 2\varepsilon_{12}^+ \; 2\varepsilon_{13}^- \; 2\varepsilon_{13}^+ \; 2\varepsilon_{23}^- \; 2\varepsilon_{23}^+ \; \bar{\varepsilon}_{33}\right]^{\mathrm{T}}$$

$$\hat{\boldsymbol{\varepsilon}} = \left[\hat{\varepsilon}_{11}^- \; \hat{\varepsilon}_{11}^+ \; \hat{\varepsilon}_{22}^- \; \hat{\varepsilon}_{22}^+ \; 2\hat{\varepsilon}_{12}^- \; 2\hat{\varepsilon}_{12}^+ \; 2\hat{\varepsilon}_{13}^- \; 2\hat{\varepsilon}_{13}^+ \; 2\hat{\varepsilon}_{23}^- \; 2\hat{\varepsilon}_{23}^+ \; \hat{\varepsilon}_{33}\right]^{\mathrm{T}}$$

$$\mathbf{H} = \left[H_{11}^- \; H_{11}^+ \; H_{22}^- \; H_{22}^+ \; H_{12}^- \; H_{12}^+ \; H_{13}^- \; H_{13}^+ \; H_{23}^- \; H_{23}^+ \; H_{33}\right]^{\mathrm{T}}, \qquad \mathbf{Q}_{\mathrm{S}}^{(\ell)} = \left[Q_{\mathrm{S}}^{(\ell)-} Q_{\mathrm{S}}^{(\ell)+}\right]^{\mathrm{T}}$$

$$\mathbf{P}_B = \left[P_1^- \; P_1^+ \; P_2^- \; P_2^+ \; P_3^- \; P_3^+\right]^{\mathrm{T}}, \qquad \mathbf{P}_S = \left[-p_1^- \; p_1^+ \; -p_2^- \; p_2^+ \; -p_3^- \; p_3^+\right]^{\mathrm{T}}$$

$$H_{\alpha\beta}^{\pm} = \sum_k \int_{d_{k-1}}^{d_k} \sigma_{\alpha\beta}^{(k)} N^{\pm} \mathrm{d}\alpha_3, \qquad H_{\alpha 3}^{\pm} = \sum_k \int_{d_{k-1}}^{d_k} \sigma_{\alpha 3}^{(k)} N^{\pm} \mathrm{d}\alpha_3 \tag{19}$$

$$H_{33} = \sum_k \int_{d_{k-1}}^{d_k} \sigma_{33}^{(k)} \mathrm{d}\alpha_3, \qquad P_i^{\pm} = \sum_k \int_{d_{k-1}}^{d_k} b_i^{(k)} N^{\pm} \mathrm{d}\alpha_3$$

$$
\mathbf{D}_{\mathrm{ME}}^{(\ell)} = \begin{bmatrix}
-n_\ell^0 e_{31}^{(\ell)} & n_\ell^0 e_{31}^{(\ell)} & 0 & 0 & 0 & 0 \\
-n_\ell^1 e_{31}^{(\ell)} & n_\ell^1 e_{31}^{(\ell)} & 0 & 0 & 0 & 0 \\
-n_\ell^0 e_{32}^{(\ell)} & n_\ell^0 e_{32}^{(\ell)} & 0 & 0 & 0 & 0 \\
-n_\ell^1 e_{32}^{(\ell)} & n_\ell^1 e_{32}^{(\ell)} & 0 & 0 & 0 & 0 \\
-n_\ell^0 e_{36}^{(\ell)} & n_\ell^0 e_{36}^{(\ell)} & 0 & 0 & 0 & 0 \\
-n_\ell^1 e_{36}^{(\ell)} & n_\ell^1 e_{36}^{(\ell)} & 0 & 0 & 0 & 0 \\
0 & 0 & m_\ell^{00} e_{15}^{(\ell)} & m_\ell^{01} e_{15}^{(\ell)} & m_\ell^{00} e_{25}^{(\ell)} & m_\ell^{01} e_{25}^{(\ell)} \\
0 & 0 & m_\ell^{10} e_{15}^{(\ell)} & m_\ell^{11} e_{15}^{(\ell)} & m_\ell^{10} e_{25}^{(\ell)} & m_\ell^{11} e_{25}^{(\ell)} \\
0 & 0 & m_\ell^{00} e_{14}^{(\ell)} & m_\ell^{01} e_{14}^{(\ell)} & m_\ell^{00} e_{24}^{(\ell)} & m_\ell^{01} e_{24}^{(\ell)} \\
0 & 0 & m_\ell^{10} e_{14}^{(\ell)} & m_\ell^{11} e_{14}^{(\ell)} & m_\ell^{10} e_{24}^{(\ell)} & m_\ell^{11} e_{24}^{(\ell)} \\
-e_{33}^{(\ell)} & e_{33}^{(\ell)} & 0 & 0 & 0 & 0
\end{bmatrix} \tag{20}
$$

$$
\mathbf{D}_{\mathrm{E}}^{(\ell)} = \begin{bmatrix}
-h_\ell^{-1} \in_{33}^{(\ell)} & h_\ell^{-1} \in_{33}^{(\ell)} & 0 & 0 & 0 & 0 \\
 & -h_\ell^{-1} \in_{33}^{(\ell)} & 0 & 0 & 0 & 0 \\
 & & k_\ell^{00} \in_{11}^{(\ell)} & k_\ell^{01} \in_{11}^{(\ell)} & k_\ell^{00} \in_{12}^{(\ell)} & k_\ell^{01} \in_{12}^{(\ell)} \\
 & & & k_\ell^{11} \in_{11}^{(\ell)} & k_\ell^{01} \in_{12}^{(\ell)} & k_\ell^{11} \in_{12}^{(\ell)} \\
 & & & & k_\ell^{00} \in_{22}^{(\ell)} & k_\ell^{01} \in_{22}^{(\ell)} \\
\text{sym.} & & & & & k_\ell^{11} \in_{22}^{(\ell)}
\end{bmatrix}
$$

$$
\mathbf{D}_{\mathrm{M}} = \begin{bmatrix}
D_{11}^{00} & D_{11}^{01} & D_{12}^{00} & D_{12}^{01} & D_{16}^{00} & D_{16}^{01} & 0 & 0 & 0 & 0 & 0 \\
 & D_{11}^{11} & D_{12}^{01} & D_{12}^{11} & D_{16}^{01} & D_{16}^{11} & 0 & 0 & 0 & 0 & 0 \\
 & & D_{22}^{00} & D_{22}^{01} & D_{26}^{00} & D_{26}^{01} & 0 & 0 & 0 & 0 & 0 \\
 & & & D_{22}^{11} & D_{26}^{01} & D_{26}^{11} & 0 & 0 & 0 & 0 & 0 \\
 & & & & D_{66}^{00} & D_{66}^{01} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & D_{66}^{11} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & D_{55}^{00} & D_{55}^{01} & D_{45}^{00} & D_{45}^{01} & 0 \\
 & & & & & & & D_{55}^{11} & D_{45}^{01} & D_{45}^{11} & 0 \\
 & & & & & & & & D_{44}^{00} & D_{44}^{01} & 0 \\
 & & & & & & & & & D_{44}^{11} & 0 \\
\text{sym.} & & & & & & & & & & D_{33}
\end{bmatrix} \tag{21}
$$

$$
D_{ab}^{pq} = \sum_k n_k^{pq} C_{ab}^{(k)}, \qquad D_{mn}^{pq} = \sum_k n_k^{pq} C_{mn}^{(k)}, \qquad D_{33} = \sum_k h_k C_{33}^{(k)} \tag{22}
$$

$$
n_k^{pq} = \int_{d_{k-1}}^{d_k} \left(N^-\right)^{2-p-q} \left(N^+\right)^{p+q} d\alpha_3, \qquad n_\ell^{p} = \frac{1}{h_\ell} \int_{d_{\ell-1}}^{d_\ell} \left(N^-\right)^{1-p} \left(N^+\right)^{p} d\alpha_3
$$

$$
k_\ell^{pq} = -\int_{d_{\ell-1}}^{d_\ell} \left(N_\ell^-\right)^{2-p-q} \left(N_\ell^+\right)^{p+q} d\alpha_3, \qquad m_\ell^{pq} = \int_{d_{\ell-1}}^{d_\ell} \left(N^-\right)^{1-p} \left(N^+\right)^{p} \left(N_\ell^-\right)^{1-q} \left(N_\ell^+\right)^{q} d\alpha_3
$$

$$
a,b = 1,2,6; \quad m,n = 4,5; \quad p,q = 0,1; \quad k = 1,2,\ldots,N; \quad \ell \in \{1,2,\ldots,N\}
$$

Finite element formulation

$$\mathbf{v} = \sum_r N_r \mathbf{v}_r \quad \Leftrightarrow \quad \mathbf{v} = \sum_{r_1, r_2} \xi_1^{r_1} \xi_2^{r_2} \mathbf{v}^{r_1 r_2}, \qquad \mathbf{v}_r = \left[u_{1r}^- \; u_{1r}^+ \; u_{2r}^- \; u_{2r}^+ \; u_{3r}^- \; u_{3r}^+ \right]^{\mathrm{T}} \tag{23}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}_{\mathrm{M}} \mathbf{V} \quad \Leftrightarrow \quad \boldsymbol{\varepsilon} = \sum_{r_1, r_2} \xi_1^{r_1} \xi_2^{r_2} \mathbf{B}_{\mathrm{M}}^{r_1 r_2} \mathbf{V}, \qquad \mathbf{V} = \left[\mathbf{v}_1^{\mathrm{T}} \; \mathbf{v}_2^{\mathrm{T}} \; \mathbf{v}_3^{\mathrm{T}} \; \mathbf{v}_4^{\mathrm{T}} \right]^{\mathrm{T}} \tag{24}$$

$$\boldsymbol{\Psi}_\ell = \mathbf{B}_{\mathrm{E}} \boldsymbol{\Phi}_\ell \quad \Leftrightarrow \quad \boldsymbol{\Psi}_\ell = \sum_{r_1, r_2} \xi_1^{r_1} \xi_2^{r_2} \mathbf{B}_{\mathrm{E}}^{r_1 r_2} \boldsymbol{\Phi}_\ell \tag{25}$$

$$\boldsymbol{\Phi}_\ell = \left[\boldsymbol{\varphi}_{\ell 1}^{\mathrm{T}} \; \boldsymbol{\varphi}_{\ell 2}^{\mathrm{T}} \; \boldsymbol{\varphi}_{\ell 3}^{\mathrm{T}} \; \boldsymbol{\varphi}_{\ell 4}^{\mathrm{T}} \right]^{\mathrm{T}}, \qquad \boldsymbol{\varphi}_{\ell r} = \left[\varphi_{\ell r}^- \; \varphi_{\ell r}^+ \right]^{\mathrm{T}}$$

$N_r(\xi_1, \xi_2)$ – bilinear shape functions of the element

$\mathbf{v}_r$, $\boldsymbol{\varphi}_{\ell r}$ – displacement and electric potential vectors of element nodes

$\mathbf{B}_{\mathrm{M}}$, $\mathbf{B}_{\mathrm{M}}^{r_1 r_2}$ – strain-displacement transformation matrices of order 11×24

$\mathbf{B}_{\mathrm{E}}$, $\mathbf{B}_{\mathrm{E}}^{r_1 r_2}$ – piezoelectric transformation matrices of order 6×8; $r = 1, 2, 3, 4$; $r_1, r_2 = 0, 1$

Assumed strain field:

$$\hat{\boldsymbol{\varepsilon}} = \sum_{r_1, r_2} \xi_1^{r_1} \xi_2^{r_2} \mathbf{Q}^{r_1 r_2} \hat{\boldsymbol{\varepsilon}}^{r_1 r_2} \qquad (26)$$

$$\hat{\boldsymbol{\varepsilon}}^{00} = \left[\hat{\varepsilon}_{11}^{-00} \; \hat{\varepsilon}_{11}^{+00} \; \hat{\varepsilon}_{22}^{-00} \; \hat{\varepsilon}_{22}^{+00} \; 2\hat{\varepsilon}_{12}^{-00} \; 2\hat{\varepsilon}_{12}^{+00} \; 2\hat{\varepsilon}_{13}^{-00} \; 2\hat{\varepsilon}_{13}^{+00} \; 2\hat{\varepsilon}_{23}^{-00} \; 2\hat{\varepsilon}_{23}^{+00} \; \hat{\varepsilon}_{33}^{00} \right]^{\mathrm{T}}$$

$$\hat{\boldsymbol{\varepsilon}}^{01} = \left[\hat{\varepsilon}_{11}^{-01} \; \hat{\varepsilon}_{11}^{+01} \; 2\hat{\varepsilon}_{13}^{-01} \; 2\hat{\varepsilon}_{13}^{+01} \; \hat{\varepsilon}_{33}^{01} \right]^{\mathrm{T}}, \quad \hat{\boldsymbol{\varepsilon}}^{10} = \left[\hat{\varepsilon}_{22}^{-10} \; \hat{\varepsilon}_{22}^{+10} \; 2\hat{\varepsilon}_{23}^{-10} \; 2\hat{\varepsilon}_{23}^{+10} \; \hat{\varepsilon}_{33}^{10} \right]^{\mathrm{T}}, \quad \hat{\boldsymbol{\varepsilon}}^{11} = \left[\hat{\varepsilon}_{33}^{11} \right]$$

$$\mathbf{Q}^{01} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}^{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}^{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (27)$$

$\mathbf{Q}^{00}$ – identity matrix of order 11×11

Assumed stress resultant field:

$$\mathbf{H} = \sum_{r_1, r_2} \xi_1^{r_1} \xi_2^{r_2} \mathbf{Q}^{r_1 r_2} \mathbf{H}^{r_1 r_2} \tag{28}$$

$$\mathbf{H}^{00} = \left[H_{11}^{-00}\ H_{11}^{+00}\ H_{22}^{-00}\ H_{22}^{+00}\ H_{12}^{-00}\ H_{12}^{+00}\ H_{13}^{-00}\ H_{13}^{+00}\ H_{23}^{-00}\ H_{23}^{+00}\ H_{33}^{00} \right]^{\mathrm{T}}$$

$$\mathbf{H}^{01} = \left[H_{11}^{-01} H_{11}^{+01} H_{13}^{-01} H_{13}^{+01} H_{33}^{01} \right]^{\mathrm{T}}, \quad \mathbf{H}^{10} = \left[H_{22}^{-10} H_{22}^{+10} H_{23}^{-10} H_{23}^{+10} H_{33}^{10} \right]^{\mathrm{T}}, \quad \mathbf{H}^{11} = \left[H_{33}^{11} \right]$$

Finite element equations:

$$\hat{\boldsymbol{\varepsilon}}^{r_1 r_2} = \left(\mathbf{Q}^{r_1 r_2} \right)^{\mathrm{T}} \mathbf{B}_{\mathrm{M}}^{r_1 r_2} \mathbf{V} \tag{29}$$

$$\mathbf{H}^{r_1 r_2} = \left(\mathbf{Q}^{r_1 r_2} \right)^{\mathrm{T}} \mathbf{D}_{\mathrm{M}} \mathbf{Q}^{r_1 r_2} \hat{\boldsymbol{\varepsilon}}^{r_1 r_2} + \left(\mathbf{Q}^{r_1 r_2} \right)^{\mathrm{T}} \mathbf{D}_{\mathrm{ME}}^{(\ell)} \mathbf{B}_{\mathrm{E}}^{r_1 r_2} \mathbf{\Phi}_{\ell} \tag{30}$$

$$\sum_{r_1, r_2} \frac{1}{3^{r_1 + r_2}} \left(\mathbf{B}_{\mathrm{M}}^{r_1 r_2} \right)^{\mathrm{T}} \mathbf{Q}^{r_1 r_2} \mathbf{H}^{r_1 r_2} = \mathbf{F}_{\mathrm{B}} + \mathbf{F}_{\mathrm{S}} \tag{31}$$

$$\sum_{r_1, r_2} \frac{1}{3^{r_1 + r_2}} \left(\mathbf{B}_{\mathrm{E}}^{r_1 r_2} \right)^{\mathrm{T}} \left(\mathbf{D}_{\mathrm{ME}}^{(\ell)\mathrm{T}} \mathbf{Q}^{r_1 r_2} \hat{\boldsymbol{\varepsilon}}^{r_1 r_2} + \mathbf{D}_{\mathrm{E}}^{(\ell)} \mathbf{B}_{\mathrm{E}}^{r_1 r_2} \mathbf{\Phi}_{\ell} \right) = \mathbf{F}_{\mathrm{Q}}^{(\ell)} \tag{32}$$

$\mathbf{F}_{\mathrm{B}}, \mathbf{F}_{\mathrm{S}}, \mathbf{F}_{\mathrm{Q}}^{(\ell)}$ – element-wise body, surface traction and electric force vectors

Governing equations

$$\begin{bmatrix} \mathbf{K}_{\mathrm{M}} & \mathbf{K}_{\mathrm{ME}}^{(\ell)} \\ \mathbf{K}_{\mathrm{ME}}^{(\ell)\mathrm{T}} & \mathbf{K}_{\mathrm{E}}^{(\ell)} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{\Phi}_{\ell} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathrm{B}} + \mathbf{F}_{\mathrm{S}} \\ \mathbf{F}_{\mathrm{Q}}^{(\ell)} \end{bmatrix} \tag{33}$$

$\mathbf{K}_{\mathrm{M}}$ – stiffness matrix; $\mathbf{K}_{\mathrm{ME}}^{(\ell)}$ – piezoelectric stiffness matrix; $\mathbf{K}_{\mathrm{E}}^{(\ell)}$ – dielectric stiffness matrix:

$$\mathbf{K}_{\mathrm{M}} = \sum_{r_1, r_2} \frac{1}{3^{r_1 + r_2}} \left(\mathbf{B}_{\mathrm{M}}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{Q}^{r_1 r_2} \left(\mathbf{Q}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{D}_{\mathrm{M}} \mathbf{Q}^{r_1 r_2} \left(\mathbf{Q}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{M}}^{r_1 r_2} \tag{34}$$

$$\mathbf{K}_{\mathrm{ME}}^{(\ell)} = \sum_{r_1, r_2} \frac{1}{3^{r_1 + r_2}} \left(\mathbf{B}_{\mathrm{M}}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{Q}^{r_1 r_2} \left(\mathbf{Q}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{D}_{\mathrm{ME}}^{(\ell)} \mathbf{B}_{\mathrm{E}}^{r_1 r_2}$$

$$\mathbf{K}_{\mathrm{E}}^{(\ell)} = \sum_{r_1, r_2} \frac{1}{3^{r_1 + r_2}} \left(\mathbf{B}_{\mathrm{E}}^{r_1 r_2}\right)^{\mathrm{T}} \mathbf{D}_{\mathrm{E}}^{(\ell)} \mathbf{B}_{\mathrm{E}}^{r_1 r_2}$$

Remark

- Element stiffness matrix $\mathbf{K}_{\mathrm{M}}$ has six zero eigenvalues
- All element matrices require only direct substitutions
- All element matrices are evaluated through analytical integration

Actuator-embedded shell analysis

$$\mathbf{K}_{\mathrm{M}}\mathbf{V} = -\mathbf{K}_{\mathrm{ME}}^{(\ell)}\mathbf{\Phi}_{\ell} \tag{35}$$

$$\mathbf{\Phi}_{\ell} = \left[\varphi_{\ell 1}^{-}\ \varphi_{\ell 1}^{+}\ \varphi_{\ell 2}^{-}\ \varphi_{\ell 2}^{+}\ \varphi_{\ell 3}^{-}\ \varphi_{\ell 3}^{+}\ \varphi_{\ell 4}^{-}\ \varphi_{\ell 4}^{+}\right]^{\mathrm{T}}$$

Numerical examples through GEX4P element

I. Bimorph cantilever beam

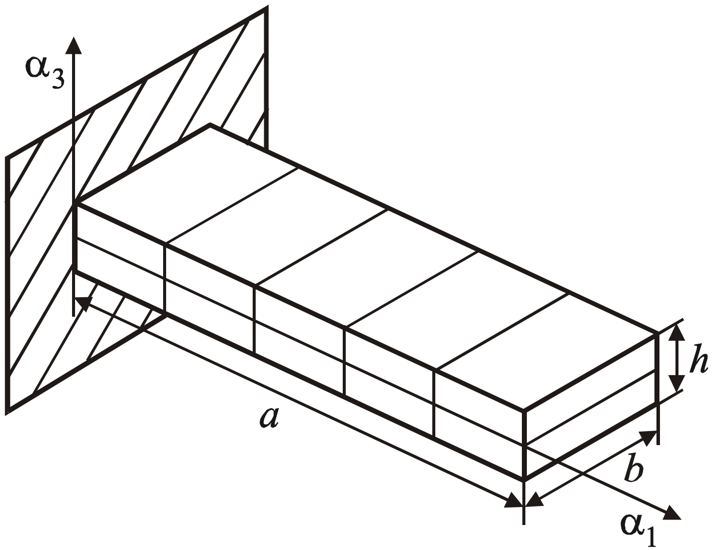

Fig. 3. Cantilever bimorph beam

Transverse midline displacement $\bar{u}_3\left(10^{-7}\,\mathrm{m}\right)$ of bimorph cantilever beam $(10\times 1$ mesh$)$

Model	Dimensionless coordinate α_1/a				
	0.2	0.4	0.6	0.8	1
Tzou (1993)	0.138	0.552	1.242	2.208	3.450
Sze et al. (2000)	0.138	0.552	1.242	2.208	3.450
Lee et al. (2003)	0.137	0.551	1.241	2.207	3.449
GEXP4 (ν=0)	0.138	0.552	1.242	2.208	3.450
GEXP4 (ν=0.29)	0.148	0.574	1.276	2.254	3.509

$a = 100\,\mathrm{mm},\ b = 5\,\mathrm{mm},\ h_1 = 0.5\,\mathrm{mm},\ h_2 = 0.5\,\mathrm{mm},\ h = 1\,\mathrm{mm},\ E = 2\,\mathrm{GPa},\ d_{31} = 23\,\mathrm{pm/V},\ d_{32} = 23\,\mathrm{pm/V}$

Problem A $(\nu = 0)$: $e_{31} = Ed_{31} = 0.046\,\mathrm{C/m^2},\quad e_{32} = e_{31}$

Problem B $(\nu = 0.29)$: $e_{31} = Ed_{31}/(1-\nu) = 0.0648\,\mathrm{C/m^2},\quad e_{32} = e_{31}$

II. Simply supported plate with PZT actuators

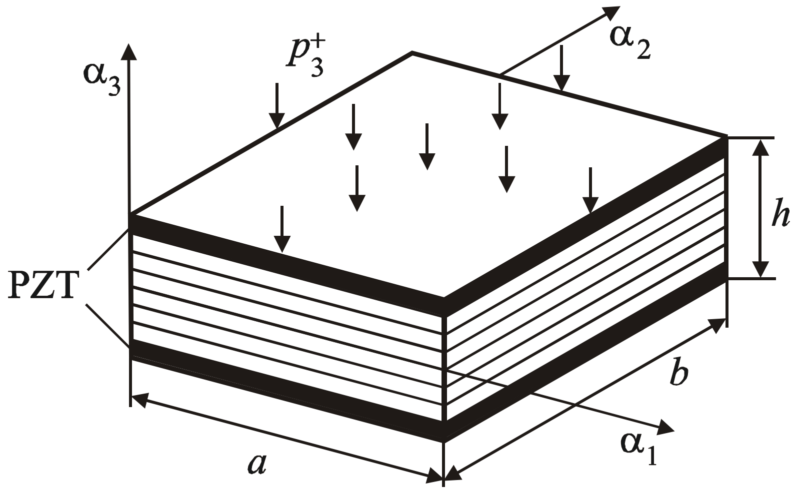

Fig. 4. Simply supported plate with PZT actuators

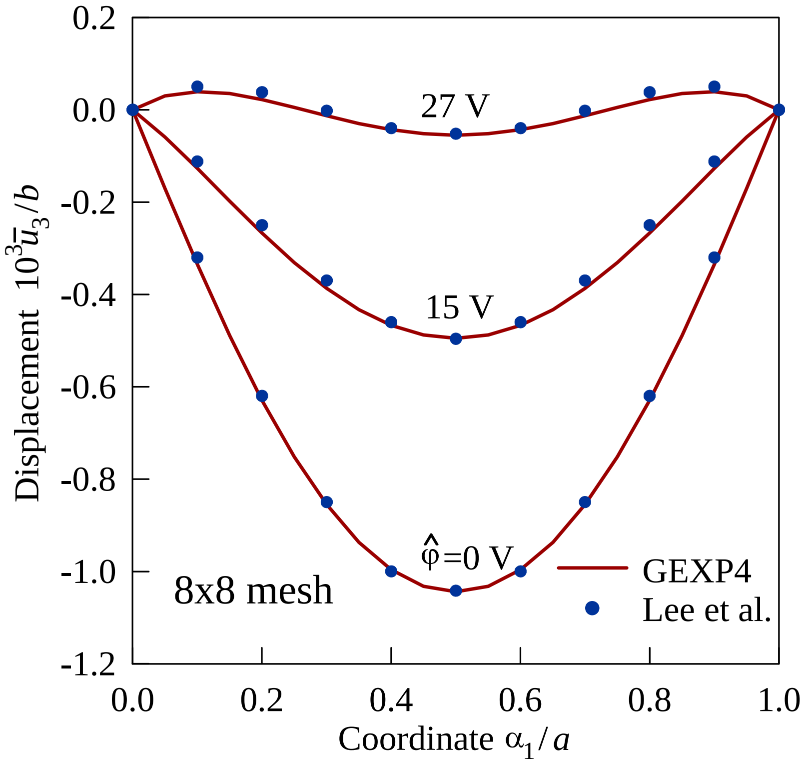

Fig. 5. Centerline midplane displacement of simply supported plate

$$a = 0.254\,\text{m},\quad b = 0.254\,\text{m},\quad h_{\text{PZT}} = 0.254\,\text{mm},\quad h_{\text{C}} = 0.828\,\text{mm},\quad h = 1.336\,\text{mm},\quad p_3^+ = -200\,\text{Pa}$$

$$\text{PZT G1195}:\ E = 63\,\text{GPa},\quad \nu = 0.3,\quad G = 24.2\,\text{GPa},\quad d_{31} = d_{32} = 254\,\text{pm/V}$$

$$\text{Graphite/Epoxy:}\ E_{11} = 150\,\text{GPa},\ E_{22} = E_{33} = 9\,\text{GPa},\ \nu_{12} = \nu_{13} = 0.3,\ G_{12} = G_{13} = 7.1\,\text{GPa},\ G_{23} = 3\,\text{GPa}$$

$$\text{Ply sequence} = [0/90/0]_s$$

III. Cantilever cylindrical shell with PZT actuators

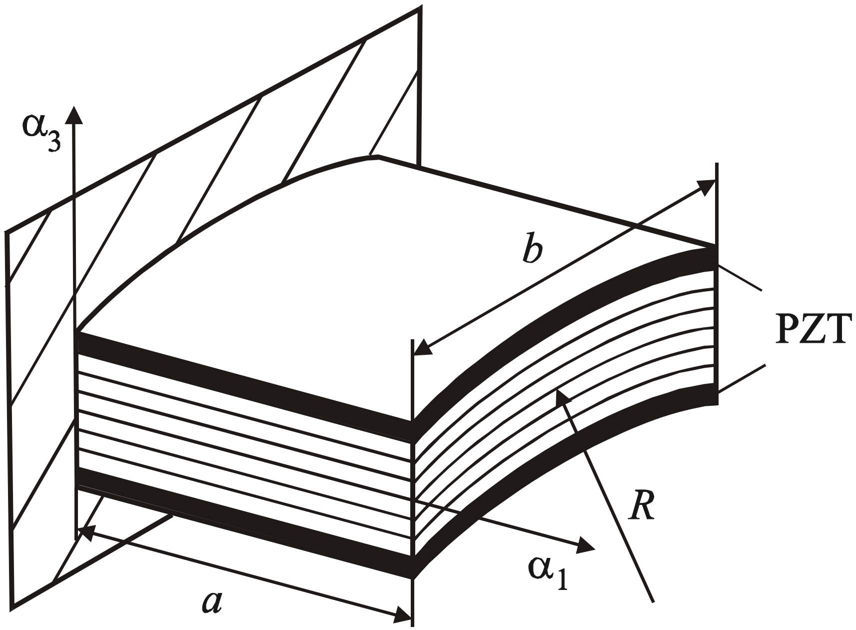

Fig. 6. Cantilever cylindrical shell with PZT actuators

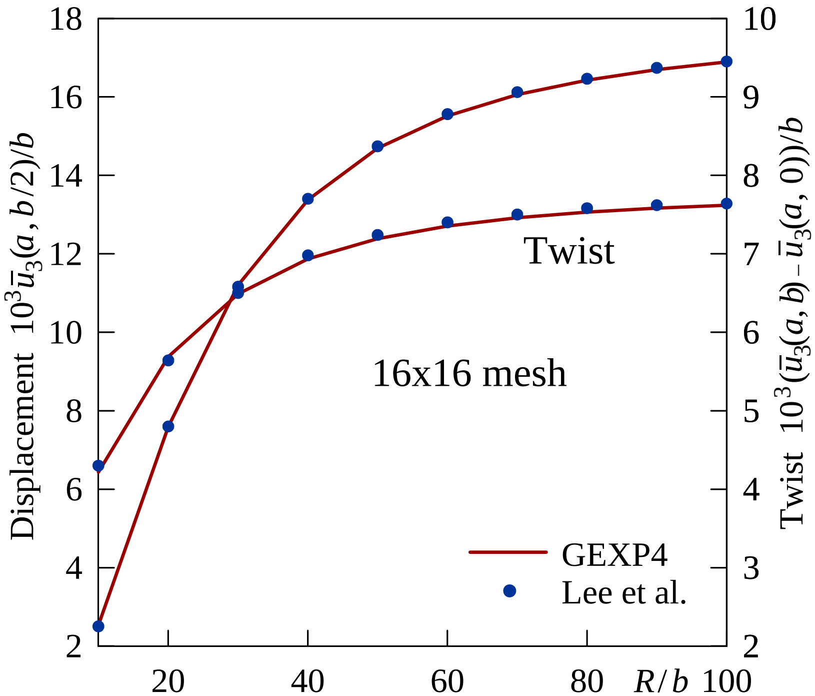

Fig. 7. Tip midsurface displacements of cantilever cylindrical shell

$$a = 0.254\,\mathrm{m}, \quad b = 0.254\,\mathrm{m}, \quad h_{\mathrm{PZT}} = 0.254\,\mathrm{mm}, \quad h_{\mathrm{C}} = 0.828\,\mathrm{mm}, \quad h = 1.336\,\mathrm{mm}$$

$$\text{Electric potential } \hat{\varphi} = 100\mathrm{V}, \text{ Ply sequence} = [30/30/0]_S$$

IV. Cantilever hyperbolic shell with PZT actuators

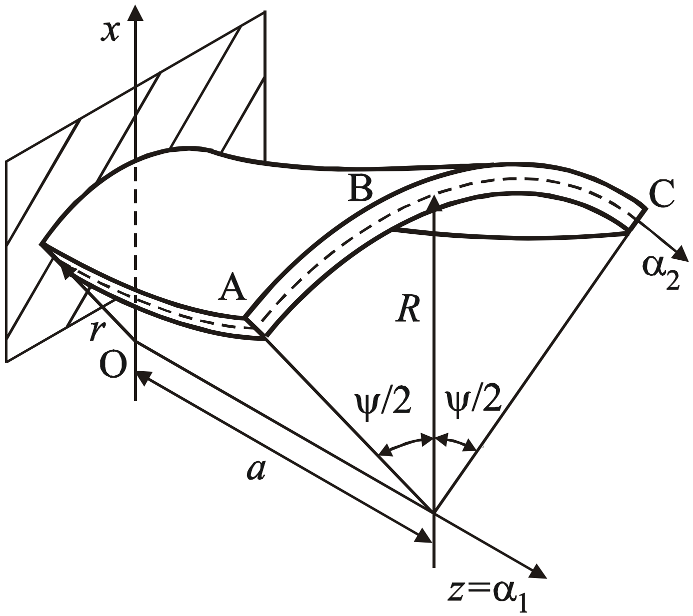

Fig. 8. Cantilever hyperbolic shell with PZT actuators

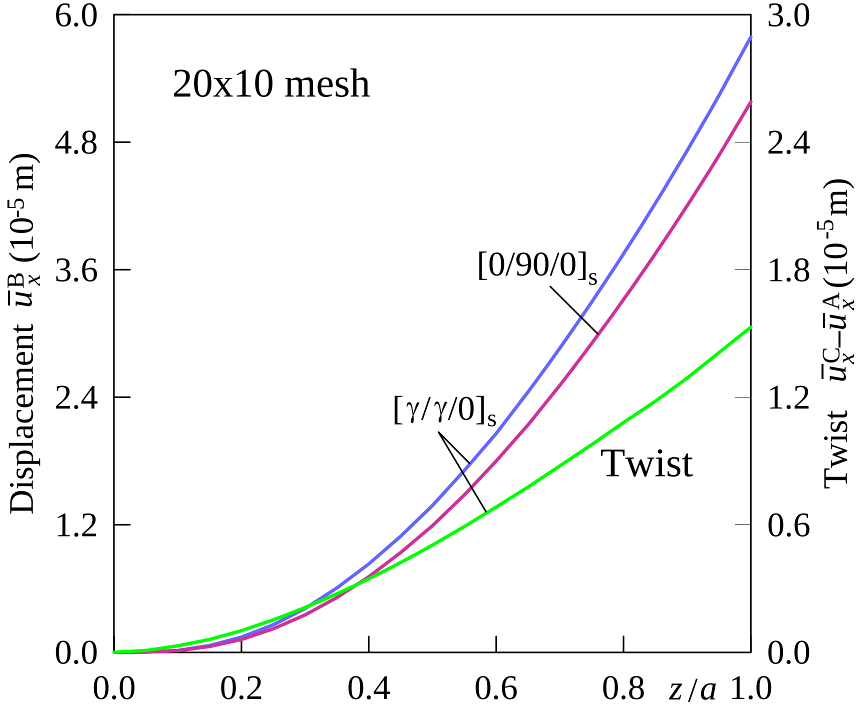

Fig. 9. Midsurface displacements of cantilever hyperbolic shell

$a = 0.254\,\text{m},\quad r = 0.254\,\text{m},\quad R = 0.508\,\text{m},\quad h_{\text{PZT}} = 0.254\,\text{mm},\quad h_{\text{C}} = 0.828\,\text{mm},\quad h = 1.336\,\text{mm},\quad \psi = 20^\circ$

Electric potential $\hat{\varphi} = 1\,\text{V}$, Ply sequence $= [0/90/0]_S$ and $[\gamma/\gamma/0]_S$

$$\cos\gamma = \frac{A_1}{\sqrt{1+\mu}}, \qquad \mu = \frac{R^2 - r^2}{a^2}, \qquad A_1 = \sqrt{1 + \frac{\mu^2 z^2}{A_2^2}}, \qquad A_2 = r\sqrt{1 + \frac{\mu z^2}{r^2}}$$

Tip midsurface displacement $\bar{u}_x \left(10^{-5}\text{m}\right)$ of cantilever hyperbolic shell

Mesh	$[0/90/0]_S$				$[\gamma/\gamma/0]_S$			
	3×1	6×2	12×4	24×8	3×1	6×2	12×4	24×8
$\bar{u}_x^A$	5.515	5.422	5.544	5.574	5.922	5.167	5.382	5.440
$\bar{u}_x^B$		5.043	5.153	5.181		5.517	5.730	5.791
$\bar{u}_x^C$	5.515	5.422	5.544	5.574	7.527	6.682	6.903	6.969

Advantages of finite element code

- Fast computation of element matrices
  - (a) No inversions are needed
  - (b) Implementation of 3D analytical integration
- Use of extremely coarse meshes