Strong and weak sampling surfaces formulations for 3D stress and vibration analyses of layered piezoelectric plates

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Abstract

This paper focuses on implementation of the sampling surfaces (SaS) method [1] for the three-dimensional (3D) stress and vibration analyses of layered piezoelectric plates. The SaS formulation is based on choosing inside the layers the arbitrary number of not equally spaced SaS parallel to the middle surface in order to introduce the displacements and electric potentials of these surfaces as basic plate unknowns. Such choice of unknowns permits the presentation of the proposed piezoelectric plate formulation in a very compact form. The SaS are located inside each layer at Chebyshev polynomial nodes that improves the convergence of the SaS method significantly. Therefore, the SaS formulation can be applied efficiently to analytical solutions for layered piezoelectric plates, which asymptotically approach the 3D exact solutions of electroelasticity as the number of SaS tends to infinity. The strong SaS formulation is based on integrating the equilibrium equations of piezoelectricity, whereas the weak SaS formulation is based on a variational approach proposed earlier by the author [2].

1 Variational SaS formulation

Consider a layered piezoelectric plate of the thickness h. Let the middle surface Ω be described by Cartesian coordinates x_1 and x_2. The coordinate x_3 is oriented in the thickness direction. According to the SaS concept, we choose inside the nth layer I_n SaS $\Omega^{(n)1}, \Omega^{(n)2}, ..., \Omega^{(n)I_n}$ parallel to the middle surface. The transverse coordinates of SaS of the nth layer located at Chebyshev polynomial nodes are written as

$$x_3^{(n)i_n} = \frac{1}{2}(x_3^{[n-1]} + x_3^{[n]}) - \frac{1}{2} h_n \cos\left(\pi \frac{2i_n - 1}{2I_n}\right), \quad (1)$$

where $x_3^{[n-1]}$ and $x_3^{[n]}$ are the transverse coordinates of interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$; $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the nth layer; the index $n = 1, 2, ..., N$ identifies the belonging of any quantity to the nth layer, where N is the number of layers; the index $i_n = 1, 2, ..., I_n$ identifies the belonging of any quantity to the SaS of the nth layer.

The through-the-thickness SaS approximations can be expressed as

$$[u_i^{(n)} \ \varepsilon_{ij}^{(n)} \ \sigma_{ij}^{(n)} \ \varphi^{(n)} \ E_i^{(n)} \ D_i^{(n)}] = \sum_{i_n} L^{(n)i_n} [u_i^{(n)i_n} \ \varepsilon_{ij}^{(n)i_n} \ \sigma_{ij}^{(n)i_n} \ \varphi^{(n)i_n} \ E_i^{(n)i_n} \ D_i^{(n)i_n}], \quad (2)$$

where $u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}, D_i^{(n)}$ are the displacements, strains, stresses, electric

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potential, electric field and electric displacements of the *n*th layer; $u_i^{(n)i_n}$, $\varepsilon_{ij}^{(n)i_n}$, $\sigma_{ij}^{(n)i_n}$, $\varphi^{(n)i_n}$, $E_i^{(n)i_n}$, $D_i^{(n)i_n}$ are the displacements, strains, stresses, electric potential, electric field and electric displacements of SaS of the *n*th layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(x_3)$ are the Lagrange basis polynomials of degree $I_n - 1$ corresponding to the *n*th layer:

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}} \quad (i_n, j_n = 1, 2, ..., I_n). \tag{3}$$

The variational SaS formulation for the laminated piezoelectric plate is based on a variational equation

$$\delta \iint_{\Omega} \sum_n \int_{x_3^{[n-1]}}^{x_3^{[n]}} \frac{1}{2}\left(\sigma_{ij}^{(n)}\varepsilon_{ij}^{(n)} - D_i^{(n)}E_i^{(n)}\right) dx_1 dx_2 dx_3 = \delta W, \tag{4}$$

where W is the work done by external electromechanical loads. Here, the summation on repeated Latin indices is implied.

2 Strong SaS formulation

For simplicity, we consider the case of linear piezoelectric materials given by

$$\sigma_{ij}^{(n)} = C_{ijkl}^{(n)}\varepsilon_{kl}^{(n)} - e_{kij}^{(n)}E_k^{(n)}, \quad D_i^{(n)} = e_{ikl}^{(n)}\varepsilon_{kl}^{(n)} + \in_{ik}^{(n)} E_k^{(n)}, \tag{5}$$

where $C_{ijkl}^{(n)}$, $e_{kij}^{(n)}$ and $\in_{ik}^{(n)}$ are the elastic, piezoelectric and dielectric constants of the *n*th layer.

The equilibrium equations and charge equation of electrostatics in the absence of body forces and free charges can be written as

$$\sigma_{ij,j}^{(n)} = 0, \quad D_{i,i}^{(n)} = 0, \tag{6}$$

where the symbol $(\ldots)_{,i}$ stands for the partial derivatives with respect to coordinates x_i.

The boundary conditions on bottom and top surfaces are defined as

$$u_i^{(1)}(-h/2) = w_i^- \text{ or } \sigma_{i3}^{(1)}(-h/2) = p_i^-, \quad \varphi^{(1)}(-h/2) = \Phi^- \text{ or } D_3^{(1)}(-h/2) = Q^-, \tag{7}$$

$$u_i^{(N)}(h/2) = w_i^+ \text{ or } \sigma_{i3}^{(N)}(h/2) = p_i^+, \quad \varphi^{(N)}(h/2) = \Phi^+ \text{ or } D_3^{(N)}(h/2) = Q^+, \tag{8}$$

where $w_i^-, p_i^-, \Phi^-, Q^-$ and $w_i^+, p_i^+, \Phi^+, Q^+$ are the prescribed displacements, surface tractions, electric potentials and electric charges at the bottom and top surfaces.

The continuity conditions at interfaces are

$$u_i^{(m)}(x_3^{[m]}) = u_i^{(m+1)}(x_3^{[m]}), \quad \sigma_{i3}^{(m)}(x_3^{[m]}) = \sigma_{i3}^{(m+1)}(x_3^{[m]}),$$

$$\varphi^{(m)}(x_3^{[m]}) = \varphi^{(m+1)}(x_3^{[m]}), \quad D_3^{(m)}(x_3^{[m]}) = D_3^{(m+1)}(x_3^{[m]}), \tag{9}$$

where the index $m = 1, 2, ..., N-1$ identifies the belonging of any quantity to the interface $\Omega^{[m]}$.

Satisfying the equilibrium equations and charge equation at inner points $x_3^{(n)m_n}$ inside the layers, the following differential equations are obtained:

$$\sigma_{i1,1}^{(n)m_n} + \sigma_{i2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n})\sigma_{i3}^{(n)i_n} = 0, \tag{10}$$

$$D_{1,1}^{(n)m_n} + D_{2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n})D_3^{(n)i_n} = 0, \tag{11}$$

where $M^{(n)i_n} = L_{,3}^{(n)i_n}$ are the derivatives of the Lagrange basis polynomials whose values at SaS $\Omega^{(n)m_n}$ are evaluated in papers [1, 2] and $m_n = 2,3,...,I_n - 1$.

Next, we satisfy the boundary conditions

$$\sum_{i_1} L^{(1)i_1}(-h/2)u_i^{(1)i_1} = w_i^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2)\sigma_{i3}^{(1)i_1} = p_i^-,$$
$$\sum_{i_1} L^{(1)i_1}(-h/2)\varphi^{(1)i_1} = \Phi^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2)D_3^{(1)i_1} = Q^-, \qquad (12)$$
$$\sum_{i_N} L^{(N)i_N}(h/2)u_i^{(N)i_N} = w_i^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2)\sigma_{i3}^{(N)i_N} = p_i^+,$$
$$\sum_{i_N} L^{(N)i_N}(h/2)\varphi^{(1)i_N} = \Phi^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2)D_3^{(N)i_N} = Q^+ \qquad (13)$$

and the continuity conditions that result in

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]})u_i^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]})u_i^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]})\sigma_{i3}^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]})\sigma_{i3}^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]})\varphi^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]})\varphi^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]})D_3^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]})D_3^{(m+1)i_{m+1}}. \qquad (14)$$

Thus, the proposed strong SaS formulation deals with $4(I_1 + I_2 + ... + I_N)$ governing equations (10)-(14) for finding the same number of SaS displacements $u_i^{(n)i_n}$ and SaS electric potentials $\varphi^{(n)i_n}$. These differential and algebraic equations have to be solved to describe the response of the layered piezoelectric plate.

3 Benchmark problems

As numerical examples, we study the static and dynamic responses of simply supported laminated piezoelectric rectangular plates. The accuracy of both SaS plate formulations is compared with each other and the Heyliger's 3D exact solutions are adopted as benchmark solutions [3, 4].

References

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