

Hybrid-mixed finite element method for piezoelectric shells through a sampling surfaces formulation

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Abstract

A hybrid-mixed exact geometry four-node piezoelectric solid-shell element using the sampling surfaces (SaS) technique is developed. The SaS formulation is based on choosing inside the n th layer I_n SaS located at Chebyshev polynomial nodes in order to introduce the displacements and electric potentials of these surfaces as basic shell variables. Such choice of unknowns with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction for displacements, strains, electric potential and electric field allows the presentation of the laminated piezoelectric shell formulation in a very compact form. The proposed hybrid-mixed four-node piezoelectric solid-shell element is based on the Hu-Washizu variational equation and exhibits a superior performance in the case of coarse meshes. It could be useful for the 3D stress analysis of thick and thin doubly-curved laminated piezoelectric shells since the SaS formulation gives the possibility to obtain numerical solutions with a prescribed accuracy, which asymptotically approach the exact solutions of electroelasticity as the number of SaS tends to infinity.

1 Hu-Washizu Variational SaS formulation

Consider a laminated shell of the thickness h . Let the middle surface Ω be described by orthogonal curvilinear coordinates θ_1 and θ_2 , which are referred to the lines of principal curvatures of its surface. The coordinate θ_3 is oriented along the unit vector normal to the middle surface. According to the SaS concept, we choose inside the n th layer I_n SaS $\Omega^{(n)1}, \Omega^{(n)2}, \dots, \Omega^{(n)I_n}$ located at Chebyshev polynomial nodes and interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ as well, where the index $n = 1, 2, \dots, N$ identifies the belonging of any quantity to the n th layer; N is the number of layers.

The through- thickness SaS approximations [1] can be written as

$$[u_i^{(n)} \ \varepsilon_{ij}^{(n)} \ \sigma_{ij}^{(n)} \ \varphi^{(n)} \ E_i^{(n)}] = \sum_{I_n} L^{(n)I_n} [u_i^{(n)I_n} \ \varepsilon_{ij}^{(n)I_n} \ \sigma_{ij}^{(n)I_n} \ \varphi^{(n)I_n} \ E_i^{(n)I_n}], \quad (1)$$

where $u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}$ are the displacements, strains, stresses, electric potential and electric field of the n th layer; $u_i^{(n)I_n}, \varepsilon_{ij}^{(n)I_n}, \sigma_{ij}^{(n)I_n}, \varphi^{(n)I_n}, E_i^{(n)I_n}$ are the displacements,

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strains, stresses, electric potential and electric field of SaS of the n th layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(\theta_3)$ are the Lagrange basis polynomials of degree $I_n - 1$ corresponding to the n th layer:

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{\theta_3 - \theta_3^{(n)j_n}}{\theta_3^{(n)i_n} - \theta_3^{(n)j_n}} \quad (i_n, j_n = 1, 2, \dots, I_n), \quad (2)$$

where the indices i_n, j_n identify the belonging of any quantity to the SaS of the n th layer.

The proposed hybrid-mixed piezoelectric solid-shell element is based on the Hu-Washizu variational equation of electroelasticity in which displacements, strains and stresses are utilized as independent variables [2]:

$$\delta \iint_{\Omega} \sum_n \int_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} \left[\frac{1}{2} \eta_{ij}^{(n)} C_{ijkl}^{(n)} \eta_{kl}^{(n)} - E_k^{(n)} e_{kij}^{(n)} \eta_{ij}^{(n)} - \frac{1}{2} E_i^{(n)} \epsilon_{ij}^{(n)} E_j^{(n)} - \sigma_{ij}^{(n)} (\eta_{ij}^{(n)} - \epsilon_{ij}^{(n)}) \right] dV = \delta W, \quad (3)$$

where $dV = A_1 A_2 (1 + k_1 \theta_3)(1 + k_2 \theta_3) d\theta_1 d\theta_2 d\theta_3$ is the infinitesimal volume element; A_1, A_2 and k_1, k_2 are the coefficients of the first fundamental form and principal curvatures of the middle surface; $\theta_3^{[n-1]}$ and $\theta_3^{[n]}$ are the transverse coordinates of interfaces; $\epsilon_{ij}^{(n)}$ and $\eta_{ij}^{(n)}$ are the displacement-dependent and displacement-independent strains; $C_{ijkl}^{(n)}, e_{kij}^{(n)}$ and $\epsilon_{ij}^{(n)}$ are the elastic, piezoelectric and dielectric constants; W is the work done by external electromechanical loads. Here, the summation on repeated Latin indices is implied.

Following the SaS technique (1), we introduce the next assumption of the hybrid-mixed solid-shell element formulation. Assume that the displacement-independent strains are distributed through the thickness of the n th layer as follows:

$$\eta_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} \eta_{ij}^{(n)i_n}, \quad (4)$$

where $\eta_{ij}^{(n)i_n}$ are the displacement-independent strains of SaS of the n th layer.

2 Finite element formulation

The finite element formulation is based on a simple interpolation of the shell via exact geometry four-node piezoelectric solid-shell elements

$$u_i^{(n)i_n} = \sum_r N_r u_{ir}^{(n)i_n}, \quad \varphi^{(n)i_n} = \sum_r N_r \varphi_r^{(n)i_n}, \quad (5)$$

where $N_r(\xi_1, \xi_2)$ are the bilinear shape functions of the element; $u_{ir}^{(n)i_n}$ and $\varphi_r^{(n)i_n}$ are the displacements and electric potentials of SaS $\Omega^{(n)i_n}$ at element nodes; ξ_1, ξ_2 are the normalized curvilinear coordinates θ_1, θ_2 ; the nodal index r runs from 1 to 4. The term "exact geometry" reflects the fact that the parametrization of the middle surface is known a priori and, therefore, the coefficients of the first and second fundamental forms are taken exactly at element nodes.

To implement the efficient analytical integration throughout the element, the enhanced ANS method is employed:

$$\epsilon_{ij}^{(n)i_n} = \sum_r N_r \epsilon_{ijr}^{(n)i_n}, \quad E_i^{(n)i_n} = \sum_r N_r E_{ir}^{(n)i_n}, \quad (6)$$

where $\varepsilon_{ijr}^{(n)i_n}$ and $E_{ir}^{(n)i_n}$ are the displacement-dependent strains and electric field of SaS of the n th layer at element nodes. The main idea of such approach can be traced back to the ANS method developed by many scientists for the isoparametric finite element formulation. In contrast with above formulation, we treat the term "ANS" in a broader sense. In the proposed exact geometry four-node solid-shell element formulation, all components of the displacement-dependent strain tensor and electric field vector are assumed to vary bilinearly throughout the biunit square in (ζ_1, ζ_2) -space.

To overcome shear and membrane locking and have no spurious zero energy modes, the robust stress and displacement-independent strain interpolations [3] are utilized:

$$\begin{bmatrix} \sigma_{11}^{(n)i_n} & \sigma_{22}^{(n)i_n} & \sigma_{33}^{(n)i_n} & \sigma_{12}^{(n)i_n} & \sigma_{13}^{(n)i_n} & \sigma_{23}^{(n)i_n} \end{bmatrix}^T = \mathbf{P} \Phi^{(n)i_n}, \quad \Phi^{(n)i_n} = [\Phi_1^{(n)i_n} \quad \Phi_2^{(n)i_n} \quad \dots \quad \Phi_{12}^{(n)i_n}]^T, \quad (7)$$

$$\begin{bmatrix} \eta_{11}^{(n)i_n} & \eta_{22}^{(n)i_n} & \eta_{33}^{(n)i_n} & \eta_{12}^{(n)i_n} & \eta_{13}^{(n)i_n} & \eta_{23}^{(n)i_n} \end{bmatrix}^T = \mathbf{P} \Psi^{(n)i_n}, \quad \Psi^{(n)i_n} = [\Psi_1^{(n)i_n} \quad \Psi_2^{(n)i_n} \quad \dots \quad \Psi_{12}^{(n)i_n}]^T, \quad (8)$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1 & \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \end{bmatrix}$$

that provides a correct rank of the element stiffness matrix.

Substituting first the through-thickness SaS approximations (1), (4) and then the finite element interpolations (5)-(8) into the Hu-Washizu variational equation (3), we arrive at the element equilibrium equations. After the elimination of column matrices $\Phi^{(n)i_n}$ and $\Psi^{(n)i_n}$ on the element level, the following system of linear equations are obtained:

$$\mathbf{K}\mathbf{U} = \mathbf{F},$$

where \mathbf{K} is the element stiffness matrix of order $12N_{\text{SaS}} \times 12N_{\text{SaS}}$; \mathbf{U} is the element displacement vector; \mathbf{F} is the element-wise surface traction vector; $N_{\text{SaS}} = \sum_n I_n - N + 1$ is the total number of SaS.

It is worth noting that the element stiffness matrix is evaluated without using the expensive numerical matrix inversion that is impossible in available isoparametric hybrid-mixed finite element formulations.

References

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