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Geometrically Exact Assumed Stress-Strain Four-Node Element Based on the 9-Parameter Shell Model

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Summary

This paper presents the geometrically exact assumed stress-strain four-node element with nine displacement degrees of freedom per node. The finite element formulation developed is based on the 9-parameter shell model by employing a new concept of interpolation surfaces (I-surfaces) inside the shell body. We introduce three I-surfaces and choose nine displacements of these surfaces as fundamental shell unknowns. Such choice allows us to represent the higher-order shell formulation in a very compact form and to derive in curvilinear reference surface coordinates the strain-displacement relationships, which are objective, i.e. invariant under all rigid-body motions.

Problem Formulation

A large number of works has been already done to develop the finite rotation higher-order shell formulation with thickness stretching. These works are devoted as a rule to the 7-parameter shell theory [1] in which the transverse normal strain varies at least linearly through the shell thickness. This fact is of great importance since the popular 6-parameter shell formulation [2-4] based on the complete 3D constitutive equations exhibits thickness locking. The errors caused by thickness locking do not decrease with the mesh refinement because the reason of stiffening lies in the shell theory itself rather than the finite element discretization.

It is well-known that a conventional way for developing the higher-order shell formulation is to utilize either quadratic or cubic series expansions in the thickness coordinate and to choose as unknowns the generalized displacements of the midsurface. Herein, the 9-parameter shell model is developed using a new concept of interpolation surfaces (I-surfaces) inside the shell body [5]. The I-surfaces are introduced in order to choose the values of displacements with correspondence to these surfaces as fundamental unknowns. Taking into account that displacement vectors of I-surfaces are resolved in the reference surface frame the proposed higher-order shell formulation is very promising for developing high performance geometrically exact solid-shell elements.

Consider a thick shell of the thickness h. As I-surfaces we choose outer surfaces Ω^- and Ω^+ and a midsurface Ω^M as well. The reference surface is assumed to be sufficiently smooth and without any singularities. Let the reference surface be referred to the orthogonal curvilinear coordinates θ^1 and θ^2 which coincide with

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lines of principal curvatures of its surface, whereas the coordinate θ^3 is oriented along the unit vector normal to the reference surface. Introduce also the following notations: $\mathbf{e}_i$ are the orthonormal basis vectors of the reference surface; A_α and k_α are the coefficients of the first fundamental form and principal curvatures of the reference surface; d^I are the distances from the reference surface to outer and middle surfaces; indices i, j range from 1 to 3; indices α, β range from1 to 2; indices I, J and A, B identify the belonging of any quantity to I-surfaces and take, respectively, the values $-$, M, $+$ and $-$, $+$.

The displacement field is approximated in the thickness direction as follows:

$$\mathbf{u} = \sum_I L^I \mathbf{u}^I, \quad \mathbf{u}^I = \sum_i u_i^I \mathbf{e}_i, \tag{1}$$

$$L^- = N^- \left(N^- - N^+\right), \quad L^M = 4N^- N^+, \quad L^+ = N^+ \left(N^+ - N^-\right), \tag{2}$$
$$N^- = \left(d^+ - \theta^3\right)/h, \quad N^+ = \left(\theta^3 - d^-\right)/h,$$

where $\mathbf{u}^I\left(\theta^1, \theta^2\right)$ are the displacement vectors of I-surfaces; $L^I\left(\theta^3\right)$ are the Lagrange polynomials of the second order such that $L^I\left(d^J\right) = 1$ for $J = I$ and $L^I\left(d^J\right) =$ 0for $J \neq I$; $N^A\left(\theta^3\right)$ are the Lagrange polynomials of the first order such that $N^{\mathrm{A}}\left(d^{\mathrm{B}}\right) = 1$ for $\mathrm{B} = \mathrm{A}$ and $N^{\mathrm{A}}\left(d^{\mathrm{B}}\right) = 0$ for $\mathrm{B} \neq \mathrm{A}$.

The strain-displacement relationships can be written in a form [5]:

$$\varepsilon_{\alpha i} = \sum_I L^I \varepsilon_{\alpha i}^I, \quad \varepsilon_{33} = \sum_A N^A \varepsilon_{33}^A, \tag{3}$$

where $\varepsilon_{ij}^I\left(\theta^1, \theta^2\right)$ are the linearized strains of I-surfaces defined as

$$2\varepsilon_{\alpha\beta}^I = \mu_\alpha^I \lambda_{\alpha\beta}^I + \mu_\beta^I \lambda_{\beta\alpha}^I, \quad 2\varepsilon_{\alpha 3}^I = \mu_\alpha^I \beta_\alpha^I + \lambda_{3\alpha}^I, \quad \varepsilon_{33}^{\mathrm{A}} = \beta_3^{\mathrm{A}}, \tag{4}$$

$$\lambda_{\alpha\alpha}^I = \frac{1}{A_\alpha} u_{\alpha,\alpha}^I + B_{\alpha\beta} u_\beta^I + k_\alpha u_3^I, \quad \lambda_{\beta\alpha}^I = \frac{1}{A_\alpha} u_{\beta,\alpha}^I - B_{\alpha\beta} u_\alpha^I \quad \text{for} \quad \beta \neq \alpha, \tag{5}$$
$$\lambda_{3\alpha}^I = \frac{1}{A_\alpha} u_{3,\alpha}^I - k_\alpha u_\alpha^I, \quad \mu_\alpha^I = 1 + k_\alpha d^I,$$
$$\beta_i^- = \frac{1}{h}\left(-3u_i^- + 4u_i^M - u_i^+\right), \quad \beta_i^M = \frac{1}{h}\left(-u_i^- + u_i^+\right),$$
$$\beta_i^+ = \frac{1}{h}\left(u_i^- - 4u_i^M + 3u_i^+\right), \quad B_{\alpha\beta} = \frac{1}{A_\alpha A_\beta} A_{\alpha,\beta}.$$

The strain-displacement relationships (3), (4) are very attractive since they represent all rigid-body motions in convected curvilinear coordinates exactly. Besides, the linear through-thickness distribution of the transverse normal strain allows utilizing the 3D constitutive equations.

The proposed finite element formulation is based on a simple and efficient approximation of shells via geometrically exact four-node solid-shell elements. The term "geometrically exact" reflects the fact that coefficients of the first and second fundamental forms are taken exactly at every integration point. Therefore, no approximation of the reference surface is needed. To avoid shear and membrane locking and have no spurious zero energy modes, the assumed strain and stress resultant fields are introduced

$$\varepsilon_{\alpha i}^{\mathrm{AS}} = \sum_I L^I E_{\alpha i}^I, \quad \varepsilon_{33}^{\mathrm{AS}} = \sum_A N^A E_{33}^A, \tag{6}$$

$$H_{\alpha i}^I = \int_{d^-}^{d^+} L^I \sigma_{\alpha i} d\theta^3, \quad H_{33}^I = \int_{d^-}^{d^+} N^A \sigma_{33} d\theta^3. \tag{7}$$

In this connection the Hu-Washizu variational principal has to be applied. Taking into account that displacement vectors of I-surfaces are represented in the reference surface frame (1), the developed finite element formulation has computational advantages compared with conventional isoparametric hybrid/mixed formulations because our elemental stiffness matrix requires only direct substitutions, i.e. no inversion is needed, and it is evaluated by using the full analytical integration [6].

Numerical Tests

The robustness of the proposed geometrically exact four-node solid-shell element GEX9P4 is assessed with several problems extracted from the literature. To substantiate this point we invoke a high performance geometrically exact four-node solid-shell element GEX6P4 [6] based on the 6-parameter shell formulation.

Consider first a rectangular homogeneous simply supported plate subjected to the sinusoidally distributed pressure load (Fig. 1) with material data $E = 10^7$ and $\nu = 0.3$. Due to symmetry of the problem, only one quarter of the plate is modeled by 32×32 mesh of GEX9P4 elements. A comparison with analytical solutions based on the elasticity theory [7] and classical plate theory (CPT) as well is given in Fig. 2.

Next we study a rectangular two-layer cross-ply simply supported plate subjected to sinusoidal loading (Fig. 1) with ply-orientation [0/90] and $E_L = 2.5 \times 10^7$, $E_T = 10^6$, $G_{LT} = 5 \times 10^5$, $G_{TT} = 2 \times 10^5$, $\nu_{LT} = \nu_{TT} = 0.25$. Owing to symmetry, one quarter of the plate is discretized again by 32×32 mesh of GEX9P4 elements. Fig. 3 displays the transverse midsurface displacement at the center point and a comparison with analytical solutions based on the elasticity theory [8] and classical plate theory.

To illustrate the capability of the developed geometrically exact four-node element to overcome membrane and shear locking phenomena and to compare it with

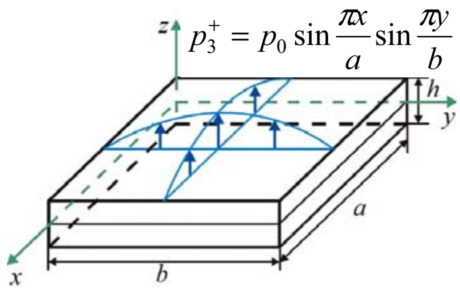

Figure 1: Rectangular plate $(a = b)$

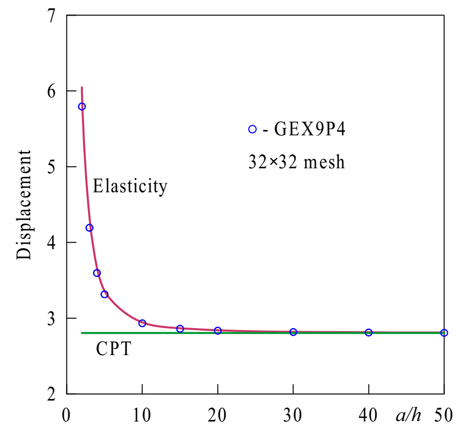

Figure 2: Transverse displacement at center point $100Eh^3u_{30}^M/p_0a^4$ of rectangular homogeneous plate.

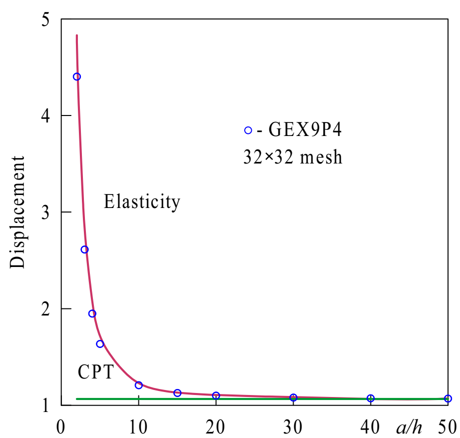

Figure 3: Transverse displacement at center point $100E_Th^3u_{30}^M/p_0a^4$ of rectangular two-layer cross-ply plate

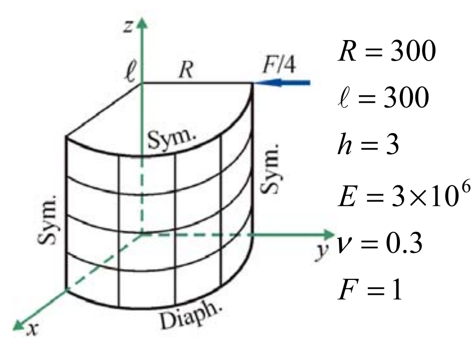

Figure 4: Pinched cylindrical shell with rigid diaphragms

high performance isoparametric four-node quadrilateral elements, we consider one of the most demanding standard linear tests, namely, a short cylindrical shell supported by two rigid diaphragms at the ends and loaded by two opposite concentrated forces in its middle section. Owing to symmetry of the problem, only one octant of the shell is modeled with regular meshes of GEX9P4 elements (Fig. 4). Table 1 lists the normalized transverse displacement under the applied load and a comparison with isoparametric elements extracted from the literature in [6]. The displacement is normalized with respect to the well-known analytical solution -1.8248×10^{-5}. As can be seen, both geometrically exact shell elements exhibit an excellent performance for coarse meshes but a GEX9P4 element is a bit stiff because of utilizing the complete 3D constitutive equations.

Table 1: Normalized transverse displacement u_3^{M} under applied load of pinched cylindrical shell

Mesh	Isoparametric elements [6]			Geometrically exact elements	
	Hughes, Liu	Bathe, Dvorkin	Simo et al.	GEX6P4 [6]	GEX9P4
4×4	0.373	0.370	0.399	0.8900	0.8691
8×8	0.747	0.740	0.763	0.9412	0.9322
16×16	0.935	0.930	0.935	0.9861	0.9826

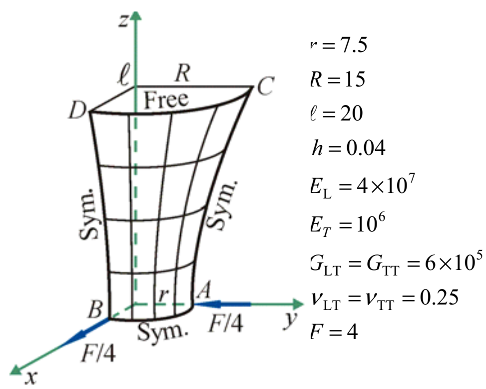

Figure 5: Pinched three-layer cross-ply hyperbolic shell with ply orientation [90/0/90]

A three-layer cross-ply hyperbolic shell subjected to two pairs of opposite concentrated forces was proposed for testing non-linear formulations for composite shells, while we employ this example as a linear benchmark test to assess the analytical integration schemes developed. Besides, as in the pinched cylinder example we can verify a proper representation of inextensional bending and, additionally, this is an excellent test for the ability of the element to model rigid-body motions. Due to symmetry of the problem, only one octant of the shell is discretized with uniform meshes of proposed elements (Fig. 5). Table 2 lists results of the convergence study through normalized midsurface displacements at points A and C. The displacements are normalized with respect to the values [6] $-u_y^{\mathrm{M}}(A) = u_x^{\mathrm{M}}(B) = 0.1013$ and $u_y^{\mathrm{M}}(C) = -u_x^{\mathrm{M}}(D) = 0.09785$, where u_x^{M} and u_y^{M} denote displacements of the midsurface in x and y directions. One can observe that developed schemes of the analytical integration perform very well.

Table 2: Normalized displacements at points A and C of pinched three-layer cross-ply hyperbolic shell

Mesh	GEX6P4 [6]		GEX9P4	
	$u_y^{\mathrm{M}}(A)$	$u_y^{\mathrm{M}}(C)$	$u_y^{\mathrm{M}}(A)$	$u_y^{\mathrm{M}}(C)$
2×2	0.5961	1.1474	0.6811	1.3119
4×4	0.8726	1.0337	0.8808	1.0221
8×8	0.9610	1.0089	0.9611	1.0087
16×16	0.9877	1.0021	0.9879	1.0022

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