NUMERICAL SOLUTION OF PROBLEMS INVOLVING THE STATICS OF GEOMETRICALLY NONLINEAR ANISOTROPIC MULTILAYER SHELLS OF REVOLUTION

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Grigolyuk and Kogan [1] shed light on the status of the theory of multilayer shells. The simplest alternate scheme of geometrically anisotropic multilayer shells is expounded upon here on the basis of a Timoshenko-type shear model. A rather complete list of studies performed in this area can be found in [2-5]. Let us point up a number of new publications [6-12], which shed light on isotropic and anisotropic, linear and nonlinear, and homogeneous and heterogeneous shells. It is apparent from this review that Timoshenko's hypothesis has been widely expanded upon in shell theory. In light of these factors, studies have brought about the development of variants to the Timoshenko-type theory.

The simplest nonlinear variant of the theory of homogeneous shells in quadratic approximation was first proposed by Marguerre [13]. Grigolyuk and Mamai [14] shed light on certain generalizations of Marguerre's theory of noninclined shells.

Basic attention was focused on study of the effect of anisotropy in multilayer shells of revolution. As compared with the universally adopted approach, the principal difference lies in the statement and method of solution of axisymmetric problems of nonlinear anisotropic laminar shells. The principal characteristic of the problem under consideration is the fact that the twisting moment, torsion of the initial surface of the shell, and other quantities characterizing the stress—strain state of the design are not zero. It should be noted that the problems under analysis have also been previously exposed within the framework of the linear theory [15-20].

In recent years, shells fashioned from orthotropic materials oriented so that the principal elastic stresses do not coincide with the directions of coordinate lines have come into widespread use in engineering. Shells of revolution made of an even number of antisymmetrically positioned anisotropic layers are investigated below. If the number of layers in the shell is sufficiently high, it is not essential to consider anisotropy. This result is well known and has been repeatedly noted in the literature, e.g., by Teters et al. [21] and Brewer [22]. In the opposite case, the effect of anisotropy should not be disregarded.

The ultimate purpose of the study was the development of an effective numerical algorithm for determination of the stress—strain state of anisotropic multilayer shells of revolution. The process of successive approximations, which was based on Newton's modified method [23], was employed for numerical solution of a solving system of nonlinear ordinary tenth-order differential equations. A linearized system of differential equations is integrated numerically by the orthogonal-sweep method, which is stable in the given class of problems.

1. Let us examine a thin multilayer shell comprised of N anisotropic layers. Let us select the inner surface of any k-th layer beyond the reference surface Ω, or the contact surface of layers, which can be assigned curvilinear orthogonal coordinates α_1, α_2. Let the coordinate z be read off in the direction of an increase in the external normal to the initial surface. The reduction in shell thickness is disregarded.

Let us introduce the necessary designations: h, total thickness of the shell, h_k, thickness of the k-th layer; δ_k, distance from the initial surface to the upper boundary of the k-th layer; A_i, Lamé's parameters; k_i, curvatures of the coordinate lines; u_i and ω, tangential displacements and normal displacement, respectively, of points on the initial surface; u_i^k, tangential displacements of points on the k-th layer; β_i and μ_i, functions characterizing transverse shear; q, normal load. Here and hereafter, i = 1, 2 and k = 1, 2, ..., N.

According to Timoshenko's kinematic hypothesis for an entire multilayer package, we have

$$u_i^k = u_i + z\beta_i.$$

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For the tangential stresses, we can take advantage of the independent approximation

$$\sigma_{i3}{}^{k}=\mu_i f(z). \tag{1.1}$$

Here f(z) is, a priori, a given function, which is continuous and satisfies the conditions $f(\delta_N) = f(\delta_0) = 0$. The independent approximation of tangential stresses introduces only a formal contraction in Timoshenko-type theory, since the elasticity relationships used for them are integrated with respect to the thickness of the package. In Eq. (1.1), the terms $1 + k_i z$ are mising in the denominator [24]. If, however, we consider the assumption made concerning the thinness of the shell walls, it is completely admissible to neglect the quantities $k_i z$ as compared to unity, the retention of which does not increase the accuracy of the final result. Thus, let us proceed to the next statement.

Let us now turn to nonlinear strain relationships [25]. In the case of the simplest nonlinear variant of the theory of laminar shells in quadratic approximation for small elongations and displacements, expressions defining the tensor of k-th-layer strains will take the form

$$\varepsilon_{ii}{}^{k}=E_{ii}+zK_{ii};\quad \varepsilon_{12}{}^{k}=E_{12}+zK_{12};\quad \varepsilon_{i3}{}^{k}=\beta_i-\theta_i;\quad \varepsilon_{33}{}^{k}=0;\quad E_{ii}=\varepsilon_i+\frac{1}{2}\theta_i{}^2;$$
$$E_{12}=\omega+\theta_1\theta_2;\quad K_{ii}=\varkappa_i;\quad K_{12}=\tau_1+k_1\omega_2+\tau_2+k_2\omega_1, \tag{1.2}$$

where

$$\theta_1=k_1u_1-\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1};\quad \omega=\omega_1+\omega_2;$$
$$\varepsilon_1=\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2+k_1w;\quad \varkappa_1=\frac{1}{A_1}\frac{\partial\beta_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\beta_2; \tag{1.3}$$
$$\omega_1=\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_1;\quad \tau_1=\frac{1}{A_1}\frac{\partial\beta_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\beta_1\quad (1\rightleftarrows 2).$$

Let us introduce specific forces and moments in the laminar shell in accordance with the expressions

$$T_1=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{11}{}^{k}(1+k_2z)\,dz;\quad T_{12}=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{12}{}^{k}(1+k_2z)\,dz;$$
$$M_1=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{11}{}^{k}z(1+k_2z)\,dz;\quad M_{12}=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{12}{}^{k}z(1+k_2z)\,dz;$$
$$Q_1=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{13}{}^{k}(1+k_2z)\,dz\quad (1\rightleftarrows 2), \tag{1.4}$$

in this case, the identity

$$T_{12}-k_2M_{21}=T_{21}-k_1M_{12} \tag{1.5}$$

follows from relationships (1.4). Proceeding from Hooke's generalized law, the stresses in the layers can be written as

$$\sigma_{11}{}^{k}=b_{11}{}^{k}\varepsilon_{11}{}^{k}+b_{12}{}^{k}\varepsilon_{22}{}^{k}+b_{16}{}^{k}\varepsilon_{12}{}^{k};\quad \sigma_{22}{}^{k}=b_{12}{}^{k}\varepsilon_{11}{}^{k}+b_{22}{}^{k}\varepsilon_{22}{}^{k}+b_{26}{}^{k}\varepsilon_{12}{}^{k};$$
$$\sigma_{12}{}^{k}=b_{16}{}^{k}\varepsilon_{11}{}^{k}+b_{26}{}^{k}\varepsilon_{22}{}^{k}+b_{66}{}^{k}\varepsilon_{12}{}^{k}. \tag{1.6}$$

Let us introduce the stresses from (1.6) in Eq. (1.4) and, taking (1.2) into account, we arrive at relationships relating specific forces and moments to deformations and curvature variations of the initial reference surface (an unconventional entry form is employed for convenience and will be used in the numerical algorithm):

$$GU_a=R;\quad U_a=[E_{11},E_{12},K_{11},K_{12},T_2,M_2]^T;$$
$$R=[T_1-A_{12}E_{22}-B_{12}K_{22};\ S-A_{26}E_{22}-B_{26}K_{22};\ M_1-B_{12}E_{22}-C_{12}K_{22};$$
$$H-B_{26}E_{22}-C_{26}K_{22};\ -A_{22}E_{22}-B_{22}K_{22},\ -B_{22}E_{22}-C_{22}K_{22}]^T; \tag{1.7}$$

$$
G=\begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} & 0 & 0 \\ A_{16} & A_{66} & B_{16} & B_{66} & 0 & 0 \\ B_{11} & B_{16} & C_{11} & C_{16} & 0 & 0 \\ B_{16} & B_{66} & C_{16} & C_{66} & 0 & 0 \\ A_{12} & A_{26} & B_{12} & B_{26} & -1 & 0 \\ B_{12} & B_{26} & C_{12} & C_{26} & 0 & -1 \end{bmatrix}.
$$

Here, new designations are introduced on the basis of identity (1.5):

$$T_{12}=S+k_2H;\; T_{21}=S+k_1H;\; M_{12}=M_{21}=H.$$

The components of the stiffness matrix can be determined from the equations

$$
\begin{gathered}
A_{nm}=\sum_{k=1}^{N}(\delta_k-\delta_{k-1})\,b_{nm}{}^k;\; B_{nm}=\frac{1}{2}\sum_{k=1}^{N}(\delta_k{}^2-\delta_{k-1}{}^2)\,b_{nm}{}^k; \\
C_{nm}=\frac{1}{3}\sum_{k=1}^{N}(\delta_k{}^3-\delta_{k-1}{}^3)\,b_{nm}{}^k \quad (n,m=1,2,6).
\end{gathered} \tag{1.8}
$$

Equations of equilibrium and the boundary conditions corresponding to them can be derived from the combined variational principal

$$\delta U=\delta A, \tag{1.9}$$

where A is the work performed by external loads, and the variation of the functional U can, after simple transformations, be written as:

$$
\begin{aligned}
\delta U=\iint_{\Omega}\Big\{ & T_1\delta E_{11}+T_2\delta E_{22}+S\delta E_{12}+M_1\delta K_{11}+M_2\delta K_{22}+H\delta K_{12}+Q_1(\delta\beta_1- \\
& -\delta\theta_1)+Q_2(\delta\beta_2-\delta\theta_2)+\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}[(\varepsilon_{13}{}^k-a_{45}{}^k\sigma_{23}{}^k-a_{55}{}^k\sigma_{13}{}^k)\delta\sigma_{13}{}^k+ \\
& +(\varepsilon_{23}{}^k-a_{44}{}^k\sigma_{23}{}^k-a_{45}{}^k\sigma_{13}{}^k)\delta\sigma_{23}{}^k]\,dz\Big\}A_1A_2\,d\alpha_1 d\alpha_2.
\end{aligned}
$$

Here $a_{44}{}^k$, $a_{55}{}^k$, and $a_{45}{}^k$ are elastic constants of the k-th layer [26]. Note that as a result of use of Hooke's generalized law, the coefficients before stress variations $\sigma_{ii}{}^k$ and $\sigma_{12}{}^k$ are, by identity, equal to zero; the terms in question do not therefore figure into expressions for δU.

Calculating the variation in the work of external forces and applying standard variational procedure, equations of equilibrium in terms of specific forces and moments

$$
\begin{gathered}
\frac{\partial(A_2T_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}T_2+\frac{\partial(A_1S)}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}S+\frac{\partial(k_1A_1H)}{\partial\alpha_2}+k_2\frac{\partial A_1}{\partial\alpha_2}H+ \\
+A_1A_2k_1N_1=0 \quad (1\rightleftarrows 2); \\
\frac{\partial(A_2N_1)}{\partial\alpha_1}+\frac{\partial(A_1N_2)}{\partial\alpha_2}-A_1A_2(k_1T_1+k_2T_2)=-A_1A_2q;\; \frac{\partial(A_2M_1)}{\partial\alpha_1}- \\
-\frac{\partial A_2}{\partial\alpha_1}M_2+\frac{\partial(A_1H)}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}H-A_1A_2Q_1=0 \quad (1\rightleftarrows 2); \\
N_1=Q_1-T_1\theta_1-S\theta_2 \quad (1\rightleftarrows 2)
\end{gathered}
$$

and additional relationships characteristic for Timoshenko-type theory

$$\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}(\varepsilon_{13}{}^k-a_{45}{}^k\sigma_{23}{}^k-a_{55}{}^k\sigma_{13}{}^k)f(z)\,dz=0 \quad (1\rightleftarrows 2;\; 4\rightleftarrows 5) \tag{1.10}$$

can be derived from (1.9).

Actually, Eq. (1.10) means that the elasticity relationships for the tangential stresses are satisfied integrally with the significance of f(z). Equations (1.10) make it possible to relate shear functions β_i with "superfluous" functions μ_i characterizing the shear stress. For this purpose, let us introduce $\varepsilon_{i3}{}^k$ from (1.2) and $\varepsilon_{i3}{}^k$ from (1.1) in (1.10), and after simple transformations, obtain the expression

$$\mu_1=\tilde{q}_{44}(\beta_1-\theta_1)-\tilde{q}_{45}(\beta_2-\theta_2) \quad (1\rightleftarrows 2;\ 4\rightleftarrows 5);\ \tilde{q}_{mn}=\frac{\tau\tau_{mn}}{\tau_{44}\tau_{55}-\tau_{45}^2}$$
$$(m, n=4, 5); \tag{1.11}$$
$$\tau=\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} f(z)\,dz;\ \tau_{mn}=\sum_{k=1}^{N} a_{mn}{}^k\int_{\delta_{k-1}}^{\delta_k} f^2(z)\,dz \quad (m, n=4,5).$$

The specific transverse forces remain to be computed. From the final equation of (1.4) and (1.1) and considering (1.11), one derives

$$Q_1=\tau\mu_1 \quad (1\rightleftarrows 2).$$

Thus, the simplest variant of the geometrically nonlinear theory of anisotropic multilayer Timoshenko-type shells is constructed.

Let us focus attention on the ultimate transition to isotropic homogeneous shells. If it is assumed that $f(z) = 3/2h[1 - (4z^2/h^2)]$, we can arrive at the relationships

$$\beta_1-\theta_1=\frac{6}{5}\frac{Q_1}{G'h} \quad (1\rightleftarrows 2),$$

which are derived in confirmity with Reissner [27]. Here G' is the transverse-shear modulus.

2. Let us now dwell in greater detail on the axisymmetric stress—strain state of anisotropic laminar shells of revolution. Since the shell will be deformed axisymmetrically, all quantities characterizing the stress—strain state of the shell will be functions of but one variable α_1. Equations (1.2) and (1.3) are therefore substantially satisfied:

$$\theta_1=k_1u_1-\frac{1}{A_1}\frac{dw}{d\alpha_1};\ \theta_2=k_2u_2;\ \varepsilon_1=\frac{1}{A_1}\frac{du_1}{d\alpha_1}+k_1w;\ \varepsilon_2=k_2w-\rho u_1;$$
$$\omega=\frac{1}{A_1}\frac{du_2}{d\alpha_1}+\rho u_2;\ K_{11}=\frac{1}{A_1}\frac{d\beta_1}{d\alpha_1};\ K_{22}=-\rho\beta_1;\ K_{12}=\frac{1}{A_1}\frac{d\beta_2}{d\alpha_1}+\rho\beta_2+ \tag{2.1}$$
$$+\frac{k_2}{A_1}\frac{du_2}{d\alpha_1}+k_1\rho u_2.$$

The additional geometric parameter $\rho = -(dA_2/d\alpha_1)/(A_1A_2)$ is introduced in Eqs. (2.1). Assuming $\beta_i = \theta_i$ and dropping nonlinear terms in all relationships, we arrive at the Kirchhoff—Love linear theory of thin elastic anisotropic shells, in particular, $K_{12} = 2k_2\omega$.

Analyzing elasticity relationships (1.7) and Eqs. (2.1), we see that they differ significantly from similar relationships of axisymmetric orthotropic shells. The special feature of the problem under investigation consists in the fact that here, one must deal with a complete system of nonlinear differential tenth-order equations, which can be written as

$$\frac{dT_1}{ds}=\rho(T_1-T_2)-k_1N_1;\ \frac{dN_1}{ds}=k_1T_1+k_2T_2+\rho N_1-q;\ \frac{dM_1}{ds}=\rho(M_1-M_2)+Q_1;$$
$$\frac{dS^*}{ds}=2\rho S^*+k_2(T_2\theta_2+S\theta_1);\ \frac{dH}{ds}=2\rho H+Q_2;\ \frac{du_1}{ds}=E_{11}-k_1w-\frac{1}{2}\theta_1^2;$$
$$\frac{dw}{ds}=k_1u_1-\theta_1;\ \frac{d\beta_1}{ds}=K_{11}; \tag{2.2}$$
$$\frac{du_2}{ds}=E_{12}-\rho u_2-\theta_1\theta_2;\ \frac{d\varepsilon_{23}}{ds}=K_{12}-\rho\varepsilon_{23}-2k_2(E_{12}-\theta_1\theta_2);\ S^*=S+2k_2H;$$
$$\frac{d}{ds}=\frac{1}{A_1}\frac{d}{d\alpha_1}.$$

Let us supplement canonical systems of differential equations (2.2) with five heterogeneous boundary conditions on each end of the closed shell of revolution:

$$T_1=T_1^* \text{ or } u_1=0;\ N_1=Q^*_1 \text{ or } w=0;\ M_1=M^*_1 \text{ or } \beta_1=0;$$
$$S^*=T^*_{12} \text{ or } u_2=0;\ H=M^*_{12} \text{ or } \varepsilon_{23}=0. \tag{2.3}$$

3. We will now explain the numerical algorithm. Let us introduce new designations for the desired quantities: $Y = [T_1, N_1, M_1, S^*, H, u_1, w, \beta_1, u_2, \varepsilon_{23}]^T$. The problem involving the reduction of the nonlinear edge problem to the solution sequence of linear problems can be solved by Newton's modified method [23]. According to this method, system of equations (2.2) can be linearized and written as

$$\frac{dY^{(n+1)}}{dx}=A_1F(x, Y^{(n)}, Y^{(n+1)}). \tag{3.1}$$

Vector F can be determined from the equations

$$F_1=\rho(Y_1^{(n+1)}-T_2^{(n+1)})-k_1Y_2^{(n+1)};\ F_2=k_1Y_1^{(n+1)}+\rho Y_2^{(n+1)}+k_2T_2^{(n+1)}-\chi q;\ F_3=\rho(Y_3^{(n+1)}-M_2^{(n+1)})+Q_1^{(n+1)};$$
$$F_4=2\rho Y_4^{(n+1)}+k_2(\theta_2^{(n)}T_2^{(n+1)}+T_2^{(n)}\theta_2^{(n+1)}-\chi T_2^{(n)}\theta_2^{(n)}+S^{(n)}\theta_1^{(n+1)}+$$
$$+\theta_1^{(n)}S^{(n+1)}-\chi\theta_1^{(n)}S^{(n)});\ F_5=2\rho Y_5^{(n+1)}+Q_2^{(n+1)}; \tag{3.2}$$
$$F_6=E_{11}^{(n+1)}-k_1Y_7^{(n+1)}-\theta_1^{(n)}\theta_1^{(n+1)}+\frac{1}{2}\chi(\theta_1^{(n)})^2;\ F_7=k_1Y_6^{(n+1)}-\theta_1^{(n+1)};$$
$$F_8=K_{11}^{(n+1)};\ F_9=E_{12}^{(n+1)}-\rho Y_9^{(n+1)}-\theta_1^{(n)}\theta_2^{(n+1)}-\theta_2^{(n)}\theta_1^{(n+1)}+\chi\theta_1^{(n)}\theta_2^{(n)};$$
$$F_{10}=K_{12}^{(n+1)}-\rho Y_{10}^{(n+1)}-2k_2(E_{12}^{(n+1)}-\theta_1^{(n)}\theta_2^{(n+1)}-\theta_2^{(n)}\theta_1^{(n+1)}+\chi\theta_1^{(n)}\theta_2^{(n)});$$
$$\alpha_1=x;\quad q_{mn}=\tau\tilde{q}_{mn}\quad (m, n=4, 5).$$

We have the relationships

$$\theta_2^{(n)}=k_2Y_9^{(n)};\quad K_{22}^{(n)}=-\rho Y_8^{(n)};\quad E_{22}^{(n)}=k_2Y_7^{(n)}-\rho Y_6^{(n)}+\frac{1}{2}(\theta_2^{(n)})^2;$$
$$S^{(n)}=Y_4^{(n)}-2k_2Y_5^{(n)};\quad \varkappa_n=q_{44}+Y_1^{(n)}; \tag{3.3}$$
$$\theta_1^{(n)}=(q_{44}Y_8^{(n)}-Y_2^{(n)}-S^{(n)}\theta_2^{(n)}-q_{45}Y_{10}^{(n)})\varkappa_n^{-1},$$

at the n-th step of the iteration process, and

$$\theta_2^{(n+1)}=k_2Y_9^{(n+1)},\quad K_{22}^{(n+1)}=-\rho Y_8^{(n+1)};$$
$$E_{22}^{(n+1)}=k_2Y_7^{(n+1)}-\rho Y_6^{(n+1)}+\theta_2^{(n)}\theta_2^{(n+1)}-\frac{1}{2}\chi(\theta_2^{(n)})^2; \tag{3.4}$$
$$S^{(n+1)}=Y_4^{(n+1)}-2k_2Y_5^{(n+1)};\quad \theta_1^{(n+1)}=[q_{44}Y_8^{(n+1)}-Y_2^{(n+1)}-S^{(n)}\theta_2^{(n+1)}-$$
$$-\theta_2^{(n)}(S^{(n+1)}-\chi S^{(n)})-q_{45}Y_{10}^{(n+1)}-\theta_1^{(n)}(Y_1^{(n+1)}-\chi Y_1^{(n)})]\varkappa_n^{-1};$$
$$Q_1^{(n+1)}=q_{44}(Y_8^{(n+1)}-\theta_1^{(n+1)})-q_{45}Y_{10}^{(n+1)};$$
$$Q_2^{(n+1)}=q_{55}Y_{10}^{(n+1)}-q_{45}(Y_8^{(n+1)}-\theta_1^{(n+1)})$$

at the n-th + 1 step. Here the parameter χ is introduced for the convenience of entry. When $\chi = 0$, homogeneous system of linear differential equations (3.1) should be integrated, and if $\chi = 1$, we arrive at a heterogeneous system.

Boundary conditions (2.3) can be written in the following manner:

$$Y_j(x_0)\gamma_j+Y_{j+5}(x_0)(1-\gamma_j)=0;\quad Y_j(x_m)\gamma_{j+5}+Y_{j+5}(x_m)(1-\gamma_{j+5})=0$$
$$(j=1,\ldots,5). \tag{3.5}$$

In Eqs. (3.5), the parameters γ_j assume the value 0.1 and define any combination of kinematic and static boundary conditions at the ends of the closed shell of revolution.

System of equations (3.1) can be solved by Godunov's orthogonal-sweep method [28]. According to this method, we must know the numerical value of the vector F as determined by relationship (3.2) for the n-th + 1 iteration at a certain point in the numerical algorithm. In this case, the sequence of computations will appear as:

a) Determine $\theta_2^{(n)}$, $K_{22}^{(n)}$, $E_{22}^{(n)}$, $S^{(n)}$, $\theta_1^{(n)}$ from Eqs. (3.3);

b) find $T_2^{(n)}$, solving system of linear algebraic equations (1.7) $GU_a^{(n)} = R^{(n)}$;

c) relationships (3.4) yield $\theta_2^{(n+1)}$, $K_{22}^{(n+1)}$, $E_{22}^{(n+1)}$, $S^{(n+1)}$, $\theta_1^{(n+1)}$, $Q_1^{(n+1)}$, $Q_2^{(n+1)}$;

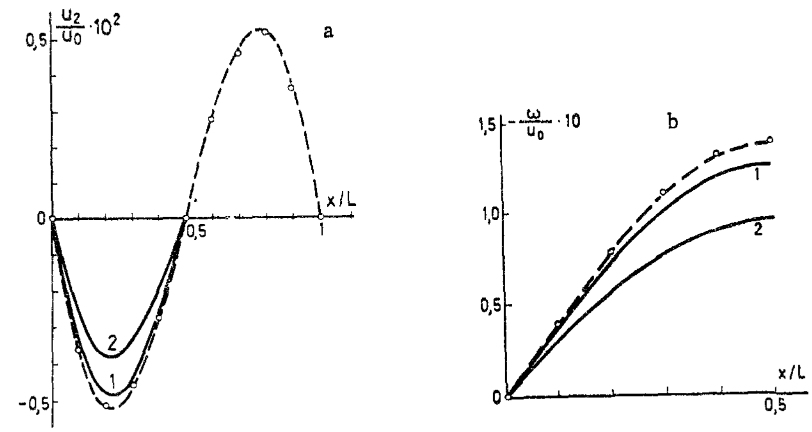

Fig. 1. Dependencies of circumferential displacement (a) and deflection (b) relative to u_0 and dimensionless axial coordinate. Nonlinear theory: u_0 = 1 (1) and 5 mm (2). Linear theory: ---) any value of u_0; ○) values after Cohen [20].

d) determine $GU_\alpha^{(n+1)} = R^{(n+1)}$ from system of algebraic equations $E_{11}^{(n+1)}$, $E_{12}^{(n+1)}$, $K_{11}^{(n+1)}$, $K_{12}^{(n+1)}$, $T_2^{(n+1)}$, $M_2^{(n+1)}$;

e) compute vector F on the right side of Eqs. (3.2). For further details of the algorithm, refer to [23], where special features of the realization of the algorithm for numerical solution of Kirchhoff–Love strength problems of orthotropic shells are presented. The algorithm in question was realized in the form of an ANSTIM program, which utilizes the matrix capability of the BÉSM-6 computer.

4. As an example of the use of ANSTIM, let us examine an anisotropic homogeneous cylindrical shell with rigidly fixed edges, one of the ends of which is located at a distance u_0. This problem was studied in a linear arrangement by Gulati and Essenburg [19] and Cohen [20]. The shell has the following geometric and mechanical characteristics [19]: h/R = 0.2, L/R = 1, $a_{11} = 0.552\cdot E$, $a_{22} = 1.076\cdot E$, $a_{66} = 1.08\cdot E$, $a_{12} = -0.0042\cdot E$, $a_{16} = -0.379\cdot E$, $a_{26} = -0.531\cdot E$, $a_{44} = a_{55} = 3.312\cdot E$, $a_{45} = 0$, and E = 0.001422 mm^2/kgf. Here L is the length of the generatrix and R is the radius of the cylindrical shell.

Relationships between the circumferential displacement and deflection relative to u_0 and a dimensionless axial coordinate are presented in Fig. 1. A comparison is made with results published recently by Cohen [20]. Let us turn our attention to the error generated by the linear theory, which increases with increasing u_0 (for nonlinear computations, R = 100 mm).

In closing, let us compute the stress–strain state of a circular torus-shape shell composed of an even number of anisotropic antisymmetrically arranged layers. It is known that these designs have come into widespread use in engineering (e.g., a pneumatic diagonal tire with an accuracy acceptable for practical computations can be referred to the class of problems under consideration [29]). The problem in question is also of interest in connection with the fact that at the present time, shells fabricated from an even number of cross-reinforced layers are computed on the basis of the theory of orthotropic shells.

The mechanical and geometric parameters of the shell can be selected close to those used in the tire industry for the production of heavy-duty diagonal tires: the elastic constants of the chord (reinforcement) $E_c = 10^4$ kgf/cm^2 and $\nu_c = 0.3$, the elastic constants of the rubber (binder) $E_r = 60$ kgf/cm^2 and $\nu_r = 0.49$, the thread diameter of the chord $d_c = 0.07$ cm, the thickness of the elementary rubber-chord layer $h_0 = 0.12$ cm, the angle formed by the chord thread with the meridian at the equator $\gamma_0 = 52°$, the chord frequency along the equator $i_0 = 9$, the number of rubber-chord layers N = 8, the internal pressure q = 5 kgf/cm^2, $R_0 = 40$ cm, and $R_1 = 10$ cm (Fig. 2). Considering the assumption concerning the thinness of the shell walls, the slope angle γ between the chord and meridian and the thread frequency i can be computed from the equation [29]

$$\sin\gamma = \frac{r}{R_0}\sin\gamma_0; \quad i = i_0\frac{R_0\cos\gamma_0}{r\cos\gamma},$$

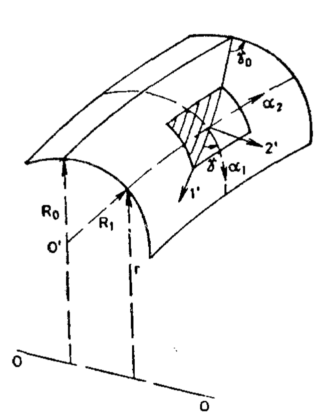

Fig. 2

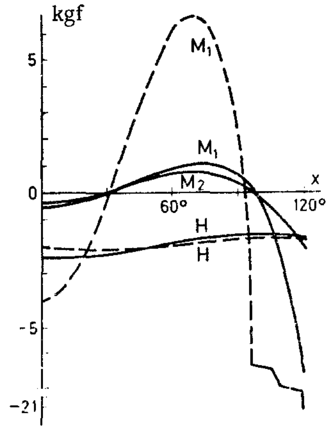

Fig. 3

Fig. 2. Computation of stress—strain state of circular torus-shape shell.

Fig. 3. Dependencies of specific bending and twisting moments on angular coordinate for nonlinear case (——) and with disregard for geometric nonlinearity (---).

where r is the distance from the axis of rotation to the shell parallel for which γ and i are determined. Note that the chord angle γ_k of the k-th layer is related to γ by the relationship $\gamma_k = (-1)^k\gamma$.

Let us take the contact surface between the fourth and fifth layers as the reference surface; Eqs. (1.8) then reduce to the simple relationship

$$(A_{11}, A_{12}, A_{22}, A_{66}) = h(b_{11}, b_{12}, b_{22}, b_{66});\quad (C_{11}, C_{12}, C_{22}, C_{66}) = \frac{h^3}{12}(b_{11}, b_{12}, b_{22}, b_{66});\quad (B_{16}, B_{26}) = \frac{h^2}{2N}(b_{16}, b_{26}). \quad (4.1)$$

The stiffness-matrix components not defined in (4.1) are considered zero. The equation cited by Teters et al. [21] were derived for the coefficients b_{mn} (m, n = 1, 2, 6). The elastic constants of the unidirectional reinforced layer can be computed in conformity with equations presented in monograph [15]. A final step is required in the interpretations, since the equations referred to are valid for reinforced plastics; this has been confirmed by experimental studies. Because there is nothing in the literature on the approach based on structural analysis of the rubber-chord layer, which would take into account the structure of the layer and the mechanical properties of the components comprising it, let us dwell on the relationships given by Malmeister et al. [15] as a first approximation.

Let x denote an angular coordinate, which varies from 0° at the equator to 120° at the rim (region of fixity). According to (3.5), a set of ten quantities $\gamma_1 = \gamma_3 = \ldots = \gamma_{10} = 0$ and $\gamma_2 = 1$ completely defines the boundary conditions at the equator and at the point on the rim. The process of successive approximations, which is used in ANSTIM exhibits rapid convergence: a relative accuracy of 10^{-5} is attained after a total of four iterations. The number of orthogonalization points was taken as 20 and 40, and had virtually no effect on the results of the computation.

Relationships between specific bending and twisting moments and the angular coordinate are shown in Fig. 3. The existence in the zone of the equator of a specific twisting moment that exceeds the specific bending moments by an order of magnitude appears somewhat surprising at first glance. Table 1, which illustrates the relationships between the layer stresses and the transverse coordinate z, fully explains the phenomenon observed. Here, the tangential stresses attain a significant magnitude, and should be considered in designing diagonal tires. Note that the linear problem does not produce such an expressed anisotropy effect owing to values of the specific moment M_1 on the extreme high side. Analysis of the last two relationships of (4.1) confirms the results obtained; it is therefore necessary to advance the theory of anisotropic shells for the design of low-ply tires. If the number of layers in the shell is sufficiently large, $\lim_{N\to\infty} B_{i6} = 0$ and traditional design methods will yield good results.

In conclusion, note that the greatest effect realized from use of the ANSTIM should be expected in the numerical solution of more complex problems involving the strength of shells of revolution fabricated from essentially anisotropic materials.

TABLE 1

Layer number	z	$x=0°$			$x=90°$		
		σ_{11}	σ_{22}	σ_{12}	σ_{11}	σ_{22}	σ_{12}
1	−0,48	44,89	62,14	45,47	39,71	20,80	26,60
	−0,45	44,74	61,94	45,32	39,99	20,94	26,79
	−0,42	44,59	61,74	45,18	40,27	21,08	26,97
	−0,39	44,44	61,54	45,03	40,55	21,22	27,16
	−0,36	44,29	61,34	44,88	40,83	21,36	27,34
2	−0,36	44,87	62,28	−45,66	39,54	20,66	−26,35
	−0,33	44,69	62,03	−45,47	39,83	20,80	−26,54
	−0,30	44,51	61,78	−45,29	40,11	20,94	−26,73
	−0,27	44,32	61,53	−45,10	40,40	21,09	−26,92
	−0,24	44,14	61,28	−44,91	40,68	21,23	−27,11
⋮	⋮		⋮			⋮	
7	0,24	41,26	57,30	41,96	46,43	24,14	31,06
	0,27	41,11	57,10	41,81	46,71	24,28	31,24
	0,30	40,96	56,90	41,66	46,99	24,42	31,43
	0,33	40,81	56,70	41,52	47,27	24,56	31,61
	0,36	40,66	56,49	41,37	47,55	24,70	31,80
8	0,36	40,52	56,28	−41,19	46,38	24,07	−30,90
	0,39	40,34	56,03	−41,01	46,66	24,21	−31,09
	0,42	40,16	55,78	−40,82	46,95	24,35	−31,28
	0,45	39,98	55,53	−40,64	47,23	24,49	−31,47
	0,48	39,80	55,28	−40,45	47,52	24,63	−31,66

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ISOPARAMETRIC TRIANGULAR FINITE ELEMENT OF A MULTILAYER SHELL AFTER TIMOSHENKO'S SHEAR MODEL

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1. STIFFNESS, MASS, AND GEOMETRIC-ELEMENT-STIFFNESS MATRICES

It is known that in the Kirchhoff–Love theory of shells, the construction of an adjusted finite element for shells of arbitrary shape is associated with significant difficulties, since in this case, it is necessary to ensure continuity between the elements of the first arbitrary deflection in formulating the problem in displacements. These difficulties do not arise in the theory of Timoshenko-type shells, since it is necessary to ensure continuity between the elements of just the most generalized displacements to construct an adjusted finite element where the principle of minimum potential energy is utilized. This makes it possible to employ the same functions of the finite-element form in the theory of Timoshenko-type shells as in elasticity.

1. Functionals That Can Be Minimized. Let us examine the derivation of stiffness, incremental-stiffness, and mass matrices for an isoparametric finite element after Timoshenko's shear model using the principle of minimum potential energy in the shell. Let us position a system of curvilinear normal coordinates $\{x^\alpha;\ x^3\}$ with a coordinate base $\{\mathbf{a}_\alpha;\ \mathbf{a}_3\}$ on the median surface of the shell so that the base vector $\mathbf{a}_3$ is directed toward the external normal to the surface. The functional of the strain energy of the shell element, which is treated as a three-dimensional body, takes the form

$$U=\frac{1}{2}\int\limits_V \sigma^{ij}e_{ij}dV=\frac{1}{2}\int\limits_V A^{ijkl}e_{kl}e_{ij}dV \quad (i, j, k, l=1, 2, 3). \tag{1.1}$$

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