THEORY AND NUMERICAL SOLUTION OF PROBLEMS OF THE STATICS OF MULTILAYERED REINFORCED SHELLS*

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The problem of stress analysis of multilayered shells can be described by the differential equations obtained in [1, 2] on the basis of the hypothesis of the broken line. Transverse shears and transverse tangential stresses on the strength of the use of Hooke's law are uniformly distributed across each layer. With the aid of independent kinematic [1] and static hypotheses the present authors constructed in [3] a geometrically nonlinear variant of the theory of shallow multilayered shells that is consistent from the point of view of the mixed variational principle and in which tangential stresses are continuous functions of the transverse coordinate, and on the boundary surfaces they assume specified values.

The present work is a further development of [3]. We deal with multilayered anisotropic shells that are not shallow. By using the mixed variational principle we obtained 2N + 3 equations of equilibrium, the boundary conditions corresponding to them, and also N + 1 integral relations of elasticity expressing the correlation between transverse tangential stresses and shears. Here, N is the number of layers in a stack. We investigated numerically the effect of nonuniformity of the tangential stresses in crosswise reinforced shells.

1. Let us examine a thin shell composed of N elastic anisotropic layers. As reference surface Ω we adopt the inner boundary surface which we attribute to the curvilinear orthogonal coordinates α_1, α_2. The transverse coordinate z will be counted toward the side of increase of the external normal to the reference surface. Let h be the thickness of the shell; h_k is the thickness of the k-th layer; δ_k is the distance between the reference surface and the upper boundary surface of the k-th layer; k_i is the curvature of the coordinate lines; A_i is the Lame parameter; u_i, w are the tangential and normal displacements, respectively, of points of the reference surface; q is the normal load; δ_{ij} is the Kronecker delta. Here, and henceforth i, j = 1, 2; k = 1, 2,..., N.

In accordance with [1] the material of each layer is transversely incompressible, and the tangential displacements within the limit of the k-th layer are linear relative to the transverse coordinate:

$$u_i^{(k)}=u_i+\sum_{n=1}^{k-1}h_n\beta_i^{(n)}+(z-\delta_{k-1})\beta_i^{(k)};\quad w^{(k)}=w. \tag{1.1}$$

Most relations of the suggested variant of the theory of multilayered anisotropic shells are formally simplified if the hypothesis (1.1) with a view to the new notation

$$\pi_{kn}=\begin{cases}h_n, & k>n\\ 0, & k\leqslant n\end{cases}\quad (n=1,2,\ldots,N)$$

is rewritten as follows:

$$u_i^{(k)}=u_i+\sum_{n=1}^{N}\pi_{kn}\beta_i^{(n)}+(z-\delta_{k-1})\beta_i^{(k)}. \tag{1.2}$$

With the aid of the limit transition $\beta_i^{(k)}=\beta_i$ hypothesis (1.2) changes into a kinematic hypothesis type Timoshenko adopted for the entire stack of layers as a whole, and this makes it possible later on to check the obtained relations by comparing them with the corresponding relations of the theory of multilayered anisotropic shells type Timoshenko [4].

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For the transverse tangential stresses we use an independent approximation. We adopt the assumption that they are distributed across the k-th layer in the form

$$\sigma_{i3}^{(k)}=f_0(z)\,\mu_i^{(0)}+f_k(z)\,\mu_i^{(k)}. \tag{1.3}$$

If we neglect the tangential loads acting on the boundary surfaces of the shell, then the a priori specified continuous functions $f_0(z)$, $f_k(z)$ have to satisfy the conditions

$$f_k(z)=0;\quad z\notin[\delta_{k-1},\delta_k];\quad f_k(\delta_{k-1})=f_k(\delta_k)=0;\quad f_0(\delta_0)=f_0(\delta_N)=0,$$

which ensures the continuity of the transverse tangential stresses along the z-coordinate everywhere in the shell including also on the layer interfaces $z=\delta_k$. Then $\sigma_{i3}^{(1)}(\delta_0)=\sigma_{i3}^{(N)}(\delta_N)=0$.

Henceforth we will use the matrix approach; this will enable us to write the cumbersome relations of the theory of multilayered anisotropic shells in a compact, easily discernible way. For instance, we represent the relations of elasticity in the form

$$\boldsymbol{\sigma}^{(k)}=\mathbf{b}^{(k)}\boldsymbol{\varepsilon}^{(k)}, \tag{1.4}$$

where $b^{(k)}$ is the matrix of the tangential rigidities of the k-th layer

$$\mathbf{b}^{(k)}=\begin{bmatrix} b_{11}^{(k)} & b_{12}^{(k)} & b_{16}^{(k)} \\ b_{12}^{(k)} & b_{22}^{(k)} & b_{26}^{(k)} \\ b_{16}^{(k)} & b_{26}^{(k)} & b_{66}^{(k)} \end{bmatrix};$$

$\sigma(k)$, $\varepsilon(k)$ are column matrices

$$\boldsymbol{\sigma}^{(k)}=[\sigma_{11}^{(k)},\sigma_{22}^{(k)},\sigma_{12}^{(k)}]^T;\quad \boldsymbol{\varepsilon}^{(k)}=[\varepsilon_{11}^{(k)},\varepsilon_{22}^{(k)},\varepsilon_{12}^{(k)}]^T.$$

2. Let us consider axisymmetric deformation of a multilayered anisotropic shell of revolution. In this case the shell will be deformed axisymmetrically, always remaining a solid of revolution, and all the magnitudes characterizing its state of stress and strain will be functions of one variable α_1 only. This, however, does not mean that the circumferential displacement u_2, the tangential stresses σ_{12}, σ_{23}, and deformations ε_{12}, ε_{23} are identically equal to zero. However, numerical experiments show that the contributuion of these magnitudes to the state of stress and strain is substantial, and when it is neglected, it sometimes leads to a distortion of the real pattern of the state of stress and strain of an anisotropic shell.

Let us now turn to the nonlinear strain relations [5]. We introduce the displacement from (1.2) into the expressions determining the strain tensor in the case of the simplest nonlinear variant of the theory of axisymmetric shells of revolution in quadratic approximation, and using the assumption that the structure is thin-walled, we obtain the formula

$$\varepsilon_{ij}^{(k)}=E_{ij}+\sum_{n=1}^{N}\pi_{kn}K_{ij}^{(n)}+(z-\delta_{k-1})K_{ij}^{(k)};\quad \varepsilon_{i3}^{(k)}=\beta_i^{(k)}-\theta_i, \tag{2.1}$$

where

$$\begin{gathered} E_{11}=\varepsilon_1+\frac{1}{2}\theta_1^2;\quad E_{12}=\omega+\theta_1\theta_2\quad(1\rightleftarrows 2);\\ \varepsilon_1=\frac{1}{A_1}\,\frac{du_1}{d\alpha_1}+k_1w;\quad \varepsilon_2=k_2w-\rho u_1;\\ \omega=\frac{1}{A_1}\,\frac{du_2}{d\alpha_1}+\rho u_2;\quad K_{11}^{(k)}=\frac{1}{A_1}\,\frac{d\beta_1^{(k)}}{d\alpha_1};\quad K_{22}^{(k)}=-\rho\beta_1^{(k)};\\ K_{12}^{(k)}=\frac{1}{A_1}\,\frac{d\beta_2^{(k)}}{d\alpha_1}+\rho\beta_2^{(k)};\quad \theta_1=k_1u_1-\frac{1}{A_1}\,\frac{dw}{d\alpha_1};\\ \theta_2=k_2u_2;\quad \rho=-\frac{1}{A_1A_2}\,\frac{dA_2}{d\alpha_1}. \end{gathered} \tag{2.2}$$

The construction of a mathematically substantiated theory of multilayered anisotropic shells within the framework of the adopted system of the independent kinematic and static hypotheses (1.2), (1.3) requires that the mixed variational principle [6] be applied:

$$\delta\Pi=\delta A^*_1+\delta A^*_2, \tag{2.3}$$

where A_1^* is the work of the external surface loads; A_2^* is the work of the external circumferential forces, and the variation of the functional Π after standard transformations in the spirit of [4, 7], with relations (1.3), (1.4), (2.1) taken into account, is represented in the form

$$\delta\Pi=\iint\limits_{\Omega}\Big\{\mathbf{T}^T\delta\mathbf{E}+\sum_{k=1}^{N}\Big[\ \mathbf{\Phi}^{(k)T}\delta\mathbf{K}^{(k)}+\mathbf{Q}^{(k)T}\delta\boldsymbol{\varepsilon}_3^{(k)}+$$
$$+\int\limits_{\delta_{k-1}}^{\delta_k}(\boldsymbol{\varepsilon}_3^{(k)}-\mathbf{a}_3^{(k)}\boldsymbol{\sigma}_3^{(k)})^T(f_0(z)\delta\boldsymbol{\mu}^{(0)}+f_k(z)\delta\boldsymbol{\mu}^{(k)})\,dz\Big]\Big\}\ A_1A_2d\alpha_1d\alpha_2. \tag{2.4}$$

In (2.4) the following notation was used:

$$\begin{gathered}\mathbf{T}=[T_{11},\ T_{22},\ T_{12}]^T;\quad \mathbf{\Phi}^{(k)}=[\Phi_{11}^{(k)},\ \Phi_{22}^{(k)},\ \Phi_{12}^{(k)}]^T;\\ \mathbf{E}=[E_{11},\ E_{22},\ E_{12}]^T;\quad \mathbf{K}^{(k)}=[K_{11}^{(k)},\ K_{22}^{(k)},\ K_{12}^{(k)}]^T;\\ \boldsymbol{\varepsilon}_3^{(k)}=[\varepsilon_{13}^{(k)},\ \varepsilon_{23}^{(k)}]^T;\quad \boldsymbol{\sigma}_3^{(k)}=[\sigma_{13}^{(k)},\ \sigma_{23}^{(k)}]^T;\\ \mathbf{Q}^{(k)}=[Q_1^{(k)},\ Q_2^{(k)}]^T;\quad \boldsymbol{\mu}^{(n)}=[\mu_1^{(n)},\ \mu_2^{(n)}]^T\quad (n=0,k);\\ \mathbf{a}_3^{(k)}=\begin{bmatrix}a_{55}^{(k)} & a_{45}^{(k)}\\ a_{45}^{(k)} & a_{44}^{(k)}\end{bmatrix}.\end{gathered} \tag{2.5}$$

Here, $a_{mn}^{(k)}$ are the transverse shear compliances of the k-th layer; T_{ij} are tangential specific forces; $Q_i^{(k)}$, $\Phi_{ij}^{(k)}$ are transverse specific forces and generalized specific moments, respectively, of the k-th layer determined by the formulas

$$\begin{gathered}T_{ij}=\sum_{k=1}^{N}T_{ij}^{(k)};\quad T_{ij}^{(k)}=\int\limits_{\delta_{k-1}}^{\delta_k}\sigma_{ij}^{(k)}dz;\quad Q_i^{(k)}=\int\limits_{\delta_{k-1}}^{\delta_k}\sigma_{i3}^{(k)}dz;\\ \Phi_{ij}^{(k)}=\int\limits_{\delta_{k-1}}^{\delta_k}\sigma_{ij}^{(k)}zdz-\delta_{k-1}T_{ij}^{(k)}+\sum_{n=1}^{N}\pi_{nk}T_{ij}^{(n)}.\end{gathered} \tag{2.6}$$

The mixed variational principle opens up the natural way of reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems of the theory of shells, thus making it possible to resolve some contradictions contained in the initial system of kinematic and static hypotheses. For instance, its use makes it possible to correlate the vectors $\beta^{(k)}=[\beta_1^{(k)},\ \beta_2^{(k)}]^T$ from (1.2) with the "superfluous" vectors $\mu^{(0)}, \mu^{(k)}$ characterizing the regularity of the distribution of the transverse tangential stresses across the thickness of the stack.

If we calculate the variation of the work of external loads and substitute the found values of δA_1^*, δA_2^* together with $\delta\Pi$ from (2.4) into the variational equation (2.3), then after the standard variational procedure, taking (2.1), (2.2), (2.5) into account, we obtain 2N + 3 equations of equilibrium in specific forces and moments

$$\begin{gathered}\frac{1}{A_1}\frac{dT_{11}}{d\alpha_1}=\rho(T_{11}-T_{22})-k_1N_1;\quad \frac{1}{A_1}\frac{dN_1}{d\alpha_1}=\rho N_1+k_1T_{11}+k_2T_{22}-q;\\ \frac{1}{A_1}\frac{d\Phi_{11}^{(k)}}{d\alpha_1}=\rho(\Phi_{11}^{(k)}-\Phi_{22}^{(k)})+Q_1^{(k)}\quad (k=1,2,\ldots,N);\\ \frac{1}{A_1}\frac{dT_{12}}{d\alpha_1}=2\rho T_{12}-k_2N_2;\\ \frac{1}{A_1}\frac{d\Phi_{12}^{(k)}}{d\alpha_1}=2\rho\Phi_{12}^{(k)}+Q_2^{(k)}\qquad (k=1,2,\ldots,N);\\ N_1=Q_1-T_{11}\theta_1-T_{12}\theta_2;\quad Q_1=\sum_{k=1}^{N}Q_1^{(k)}\qquad (1\rightleftarrows 2),\end{gathered} \tag{2.7}$$

the boundary conditions corresponding to them and the additional integral relations

$$\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}(\mathbf{\varepsilon}_3^{(k)}-\mathbf{a}_3^{(k)}\mathbf{\sigma}_3^{(k)})f_0(z)\,dz=0; \tag{2.8}$$

$$\int_{\delta_{k-1}}^{\delta_k}(\mathbf{\varepsilon}_3^{(k)}-\mathbf{a}_3^{(k)}\mathbf{\sigma}_3^{(k)})f_k(z)\,dz=0, \tag{2.9}$$

whose mechanical meaning consists in the following. The relations of elasticity for the transverse tangential stresses are fulfilled integrally across the thickness of the k-th layer with the weight function $f_k(z)$, and additionally across the thickness of the stack with the weight function $f_0(z)$.

Formulas (2.8), (2.9) are of great importance in the variant of the theory of multilayered anisotropic shells involved because with their aid in particular is it possible to correlate the vectors $\beta^{(k)}$ with $\mu^{(0)}$, $\mu^{(k)}$. For that purpose we introduce $\sigma_{i3}^{(k)}$ from (1.3) into relations (2.8), (2.9), and introducing the notation

$$\xi_k=\int_{\delta_{k-1}}^{\delta_k}f_0(z)\,dz;\quad \lambda_k=\int_{\delta_{k-1}}^{\delta_k}f_0^2(z)\,dz;$$

$$\lambda_{k0}=\int_{\delta_{k-1}}^{\delta_k}f_0(z)f_k(z)\,dz;\quad \lambda_{kk}=\int_{\delta_{k-1}}^{\delta_k}f_k^2(z)\,dz,$$

we obtain the system of linear algebraic equations with respect to the functions $\mu_i^{(0)}$, $\mu_i^{(k)}$:

$$\sum_{k=1}^{N}(\lambda_k\mathbf{a}_3^{(k)}\mathbf{\mu}^{(0)}+\lambda_{k0}\mathbf{a}_3^{(k)}\mathbf{\mu}^{(k)})=\sum_{k=1}^{N}\xi_k\mathbf{\varepsilon}_3^{(k)};\quad \lambda_{k0}\mathbf{a}_3^{(k)}\mathbf{\mu}^{(0)}+\lambda_{kk}\mathbf{a}_3^{(k)}\mathbf{\mu}^{(k)}=\mathbf{\varepsilon}_3^{(k)}. \tag{2.10}$$

Here we adopted

$$\int_{\delta_{k-1}}^{\delta_k}f_k(z)\,dz=1.$$

Taking the new designations (m, n = 4, 5)

$$q^*_{mn}=\frac{\tau_{mn}}{\tau_{44}\tau_{55}-\tau_{45}^2};\quad \tau_{mn}=\sum_{k=1}^{N}\left(\lambda_k-\frac{\lambda_{k0}^2}{\lambda_{kk}}\right)a_{mn}^{(k)};$$

$$\eta_{mn}^{(k)}=\frac{1}{\lambda_{kk}}\,\frac{a_{mn}^{(k)}}{a_{44}^{(k)}a_{55}^{(k)}-(a_{45}^{(k)})^2};$$

$$\varphi_i=\sum_{k=1}^{N}\left(\xi_k-\frac{\lambda_{k0}}{\lambda_{kk}}\right)\varepsilon_{i3}^{(k)}$$

into account, we represent the solution of the system (2.10) in the form

$$\mu_1^{(0)}=q^*_{44}\varphi_1-q^*_{45}\varphi_2;\quad \mu_1^{(k)}=\eta_{44}^{(k)}\varepsilon_{13}^{(k)}-\eta_{45}^{(k)}\varepsilon_{23}^{(k)}-\frac{\lambda_{k0}}{\lambda_{kk}}\mu_1^{(0)}$$

$$(1\rightleftarrows 2;\ 4\rightleftarrows 5). \tag{2.11}$$

Basically, formulas (2.11) solve the posed problem since the transverse tangential stresses from (1.3), the transverse specific forces from (2.6) can be expressed through the components of the vector of generalized displacements u_i, w, $\beta_i^{(k)}$. Thus we constructed the geometrically nonlinear variant of the theory of multilayered anisotropic shells which is consistent from the point of view of the mixed variational principle and takes the nonuniform distribution of the transverse tangential stresses across the stack into account.

Let us now revert to the specific forces and moments. If we substitute the stresses from (1.4) into formulas (2.5), (2.6), and integrate, taking the deformation expressions (2.1) into account, we obtain relations correlating the specific tangential forces and generalized moments with the kinematic characteristics of the reference surface. We write them in matrix form:

$$\mathbf{T}=\mathbf{A}\mathbf{E}+\sum_{k=1}^{N}\mathbf{D}^{(k)}\mathbf{K}^{(k)};\quad \mathbf{\Phi}^{(k)}=\mathbf{D}^{(k)}\mathbf{E}+\sum_{n=1}^{N}\mathbf{F}^{(kn)}\mathbf{K}^{(n)}.$$

For the rigidity matrix of the shell the following formulas apply:

$$\mathbf{A}=\sum_{k=1}^{N}\mathbf{A}^{(k)};\quad \mathbf{D}^{(k)}=\mathbf{B}^{(k)}-\delta_{k-1}\mathbf{A}^{(k)}+\sum_{n=1}^{N}\pi_{nk}\mathbf{A}^{(n)};$$

$$\mathbf{F}^{(kn)}=\delta_{kn}[\mathbf{C}^{(n)}-\delta_{n-1}\mathbf{B}^{(n)}-\delta_{n-1}(\mathbf{B}^{(n)}-\delta_{n-1}\mathbf{A}^{(n)})]+\pi_{kn}(\mathbf{B}^{(k)}-\delta_{k-1}\mathbf{A}^{(k)})+\pi_{nk}(\mathbf{B}^{(n)}-\delta_{n-1}\mathbf{A}^{(n)})+\sum_{m=1}^{N}\pi_{mk}\pi_{mn}\mathbf{A}^{(m)},$$

where $A^{(k)}$, $B^{(k)}$, $C^{(k)}$ are the matrices of membrane, membrane-flexural, and flexural rigidity of the k-th layer, respectively. If we assume that the mechanical characteristics within the limits of each layer of the shell do not depend on the transverse coordinate, we can write the expressions for these matrices in the form

$$\mathbf{A}^{(k)}=(\delta_k-\delta_{k-1})\mathbf{b}^{(k)};\quad \mathbf{B}^{(k)}=\frac{1}{2}(\delta_k^2-\delta_{k-1}^2)\mathbf{b}^{(k)};$$

$$\mathbf{C}^{(k)}=\frac{1}{3}(\delta_k^3-\delta_{k-1}^3)\mathbf{b}^{(k)}.$$

3. We will now derive the normal system of ordinary differential equations which is resolvent and fully determines the state of stress and strain of the shell. The first 2N + 3 equations were obtained earlier. These are equations of equilibrium in specific forces and moments (2.7). The other group from the 2N + 3 equations follows from the deformational relations (2.2) and can be written in the form

$$\begin{gathered}\frac{1}{A_1}\frac{du_1}{d\alpha_1}=E_{11}-k_1w-\frac{1}{2}\theta_1^2;\quad \frac{1}{A_1}\frac{dw}{d\alpha_1}=k_1u_1-\theta_1;\\ \frac{1}{A_1}\frac{d\beta_1^{(k)}}{d\alpha_1}=K_{11}^{(k)}\quad (k=1,2,\ldots,N);\\ \frac{1}{A_1}\frac{du_2}{d\alpha_1}=E_{12}-\rho u_2-\theta_1\theta_2;\\ \frac{1}{A_1}\frac{d\beta_2^{(k)}}{d\alpha_1}=K_{12}^{(k)}-\rho\beta_2^{(k)}\quad (k=1,2,\ldots,N).\end{gathered}\tag{3.1}$$

We represent the obtained system of equations (2.7), (3.1) in matrix form. For that purpose, we introduce the vector of solutions with the dimensionality 4N + 6:

$$\begin{gathered}\mathbf{Y}=[T_{11},\ N_1,\ \Phi_{11}^{(1)},\ldots,\Phi_{11}^{(N)},\ T_{12},\ \Phi_{12}^{(1)},\ldots,\Phi_{12}^{(N)},\\ u_1,\ w,\ \beta_1^{(1)},\ \ldots,\ \beta_1^{(N)},\ u_2,\ \beta_2^{(1)},\ \ldots,\ \beta_2^{(N)}]^T.\end{gathered}\tag{3.2}$$

With a view to the notations (3.2) the normal system of equations can be written as follows:

$$\frac{1}{A_1}\frac{d\mathbf{Y}}{d\alpha_1}=\mathbf{G}(\alpha_1,\mathbf{Y}).\tag{3.3}$$

We supplement the normal system (3.3) with 2N + 3 boundary conditions on each end face of the closed shell of revolution

$$\begin{gathered}Y_n(\alpha^*_1)l_n+Y_{2N+3+n}(\alpha^*_1)(1-l_n)=0;\\ Y_n(\alpha^{**}_1)l_{2N+3+n}+Y_{2N+3+n}(\alpha^{**}_1)(1-l_{2N+3+n})=0.\end{gathered}\tag{3.4}$$

In (3.4) the criteria of the boundary conditions l_n, l_{2N+3+n} (n = 1, 2 ..., 2N + 3) assume the values 0, 1 and determine an arbitrary combination of kinematic and static boundary conditions on the end faces of the shell $\alpha_1 = \alpha_1^*$, $\alpha_1 = \alpha_1^{**}$.

4. In practice the solution of nonlinear boundary-value problems is usually effected with the aid of various iteration methods. Promising are those iteration processes which at each step lead to the solution of linear boundary-value problems. Here we will solve the problem of reducing a nonlinear boundary-value problem to a sequence of linear boundary-value problems by the method of quasilinearization [8]. The method of quasilinearization proved its worth in the solution of geometrically nonlinear problems of classical Kirchhoff–Love

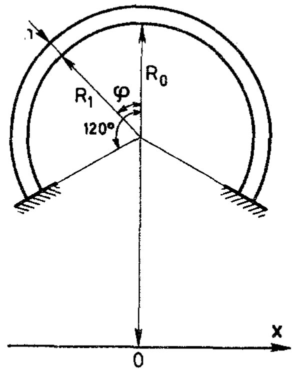

Fig. 1

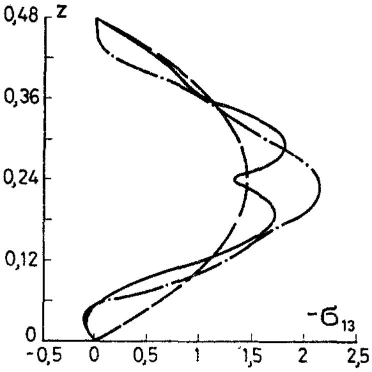

Fig. 2

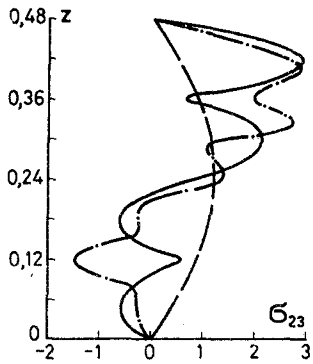

Fig. 3

shells [9, 10] and of shells type Timoshenko [4, 7]. Therefore, without entering into details of a computing nature, we present the linearized system of differential equations

$$\frac{1}{A_1}\frac{d\mathbf{Y}^{[m+1]}}{d\alpha_1}=\mathbf{G}^*(\alpha_1, \mathbf{Y}^{[m]}, \mathbf{Y}^{[m+1]}). \tag{4.1}$$

In view of the limited length of the article we omit the expressions for the vector components on the right-hand sides of the system (4.1); for details we refer the reader to the articles [4, 7] which dealt with similar methods of numerically solving nonlinear problems of multilayered anisotropic shells.

We will briefly formulate the algorithm for solving the boundary-value problem (3.3), (3.4). We begin with the trial solution $Y_n^{[0]} = 0$ ($n = 1, 2, \ldots, 4N + 6$), then we find the successive approximations $Y^{[1]}$, $Y^{[2]}$, ... by solving the linear boundary-value problems (4.1), (3.4) at each step of the iteration process by the method of orthogonal matching [11]. The choice of the zeroth initial approximation makes it possible at the first step of the successive approximations to determine the vector of the solutions $Y^{[1]}$ describing the state of stress and strain of the geometrically linear shell.

The above-explained algorithm for the numerical solution of the nonlinear boundary-value problem (3.3), (3.4) was realized in the form of standard procedures in the algorithmic language PL/1. All the numerical calculations were carried out on an ES-1060 computer.

5. Let us consider a reinforced shell made of an even number of asymmetrically arranged layers. We take it that all the layers of the shell have the same kind of structure and differ from each other only by the angle of the reinforcement $\gamma^{(k)} = (-1)^{k-1}\gamma_a$ where γ_a is a constant magnitude.

We realize the problem numerically for a four-layer circular toroidal shell (Fig. 1) with the geometric parameters $h = 0.48$ cm; $R_0 = 25$ cm; $R_1 = 5$ cm (R_0 is the distance between the axis of rotation and the equator, R_1 is the radius of the generatrix of the circle of the reference surface) made of a rubber–cord composite. The initial material of the unidirectional reinforced layer with thickness $h_0 = 0.12$ cm is textile cord with modulus of elasticity E_a =

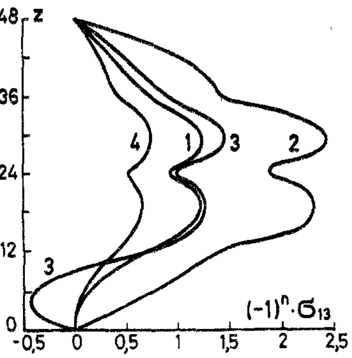

Fig. 4

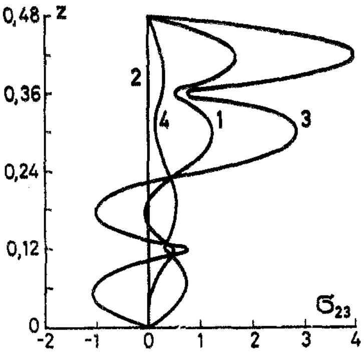

Fig. 5

$1.6 \cdot 10^5$ N/cm^2; Poisson ratio $\nu_a = 0.4$, and rubber with $E_r = 360$ N/cm^2; $\nu_r = 0.49$. The thread diameter of the cord is $d_a = 0.07$ cm; the reinforcement frequency is $i_a = 9.9$ threads/cm; $\gamma_a = 52°$. The method of calculating the elastic constants of a unidirectionally reinforced layer and some other problems of the mechanics of reinforced materials can be found in [12].

Let the shell be loaded by internal pressure $q = 15$ N/cm^2. In the numerical calculations we will assume that on the equator ($\varphi = 0°$) the condition of symmetry is fulfilled, and the section of the shell with the coordinate $\varphi = 120°$ is taken to be rigidly constrained.

The numerical results represented by the solid curves in Figs. 2-5 were obtained by integration of the normal system of ordinary differential equations of 22nd order. For the sake of comparison Figs. 2 and 3 present the results of the solution of an analogous problem by the finite element method [13] where the equations of the nonlinear theory of elasticity (dot-dash lines) were used. The dashed lines correspond to calculations on the basis of the theory of Timoshenko-type shells [4]. The graphs of the tangential stresses were plotted for the cross section of the shell with the coordinate $\varphi = 90°$. We see that the stresses σ_{13} are distributed almost parabolically across the thickness of the stack, but within the limits of an internal layer we find a considerable deviation from the regularity of a quadratic parabola. On the whole it may be said that for evaluating the strength of crosswise reinforced structures the results of the calculations of the stresses σ_{13} on the basis of Timoshenko-type shells are perfectly satisfactory. A different matter are the tangential stresses σ_{23}. Here the graph of σ_{23} (see Fig. 3), whose maximum is shifted toward the middle surface of the outer layer of the shell, is of a fairly complex nature indicating nonuniform distribution of the stresses σ_{23} across the thickness of the stack. The order of magnitude of the values of σ_{13} and σ_{23} is the same, and this in turn indicates that the effect of anisotropy makes a substantial contribution to the overall pattern of the state of stress and strain of the shell. Additional information on the nature of the distribution of the transverse tangential stresses can be obtained from Figs. 4, 5 where the curves near which the values n = 1, 2, 3, 4 are situated correspond to the calculations of shells with reinforcement angles $\gamma_a = 45, 0, 60, 30°$, respectively.

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STATE OF STRESS AND STRAIN OF HEAT-SENSITIVE CYLINDRICAL SHELLS MADE OF COMPOSITE MATERIALS*

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In the stress analysis of structures made of new fiber-reinforced materials which have low shear strength, dangerous stresses are in the first place transverse shear stresses [1]. From the point of view of practice it is important to be able to determine as accurately as possible the distribution of tangential stresses over the thickness of the shell. It is known that the temperature, too, has a strong effect of the mechanical characteristics of structural materials [2]. As a result of experimental investigations it was established that with rising temperature the shear modulus decreases more than the modulus of elasticity [3]. If a plate or shell is exposed to the effect of a stationary temperature field, and its characteristics depend on the temperature, then the shell is inhomogeneous, and this inhomogeneity is asymmetric relative to its middle surface. When a symmetric regularity of change of transverse tangential stresses across the thickness of the shell is chosen, then some error from the point of thermoelasticity is deliberately admitted, and this regularity is to some extent in contradiction to the mechanics of deformation [4].

The construction of equations of equilibrium of heat-sensitive plates and shells by replacing the kinematic and static hypotheses by the notion of a state of homogeneous stress and strain of a thin-walled element of a laminated structure [5, 6] makes it possible to eliminate some shortcomings of the refined variants of the applied theories of laminated plates and shells based on the method of hypotheses [7-10]. Such a refined model does not require the assumption of symmetric distribution of transverse tangential stresses relative to the middle surface, and it makes it possible to determine their distribution across the thickness of the shell from the solution of the obtained system of equations.

Initial prerequisites. We deal with a laminated cylindrical shell with thickness 2h, composed symmetrically or asymmetrically of N orthotropic layers and belonging to the triorthogonal system of coordinates x_1, x_2, x_3 whose principal coordinate surface coincides with the middle surface of the shell. The principal axes of orthotropy coincide with the axes of coordinates.

We assume that the shell is exposed to the effect of nonuniform steady heating of the form

$$\theta^{(k)}=\theta_0{}^{(k)}(x_1,x_2)+\theta_1{}^{(k)}(x_1,x_2)x_3. \tag{1}$$

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