GENERAL DIRECTION OF DEVELOPMENT OF THE THEORY OF MULTILAYERED SHELLS

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It has been more than 15 years since the study [1] outlined the main directions of development of the theory of multilayered shells. The survey [2] analyzed investigations conducted mainly during the 1978-1981 period, but it focused principally on applied methods of design of plates and shells made of composites and questions relating to structural design. Thus, there has been a longstanding need to examine research in the theory of multilayered shells from a common viewpoint to discern the general direction of development of the theory [1], with most of the relevant studies having been published since 1972.

We will look at investigations in which different types of nonclassical hypotheses were used to describe the mechanical behavior of each shell layer. As is known, the simplest of these hypotheses is the Timoshenko kinematic hypothesis (hypothesis of a straight line). With this approach, the order of the resolvent system of differential equations depends on the number of layers, which makes it possible to study subtle effects connected with the local character of deformation of individual shell layers. Nevertheless, a similar direction in the mechanics of heterogeneous structures – the theory of shells made of alternating stiff and flexible layers [3] – is outside the scope of this survey because the classical Kirchhoff-Love hypotheses are adopted here for the stiff layers.

We did not have the goal of discussing all of the studies we know of in the general theory of multilayered shells. Instead, we sought to offer a more complete analysis of the main approaches to the construction of this theory and to establish the interrelationship between these approaches.

1. The principles of the construction of the theory of multilayered shells were expounded in fundamental works concerning three-layer shells [4, 5], where the hypothesis of a broken line was first formulated. The latter makes it methodologically possible to construct the theory of three-layer shells in the spirit of the theory of one-layer shells.

The studies [6-9] developed the theory of multilayered shallow shells, in which the kinematic hypothesis of Timoshenko (the broken-line hypothesis) was adopted for each layer in deriving the equilibrium equations. Such concepts as a load-bearing (stiff) layer and filler (flexible layer), commonly used in the mechanics of deformable solids, lose their significance in the Grigolyuk–Chukov theory of multilayered shells. From the viewpoint of this theory, all of the layers of a shell are equivalent. This approach greatly facilitates construction of the algorithm [10-12].

Let us return to the theory being analyzed [6, 7]. In accordance with the broken-line hypothesis, the tangential displacements are determine from the formula

$$u_i^{(k)} = u_i + \sum_{n=1}^{k-1} h_n \beta_i^{(n)} + (z - \delta_{k-1}) \beta_i^{(k)}. \tag{1.1}$$

Here and below, u_i are the tangential displacements of the inside surface of the shell; h_k is the thickness of the k-th layer; h is the thickness of the shell; δ_k is the distance from the reference surface to upper boundary surface of the k-th layer ($\delta_0 = 0$); $\beta_i^{(k)}$ are angles of rotation of the normal in the k-th layer; z is the transverse coordinate; N is the number of layers; $k = 1, 2, \ldots, N$; $i = 1, 2$. Transverse contraction of the layers over the thickness

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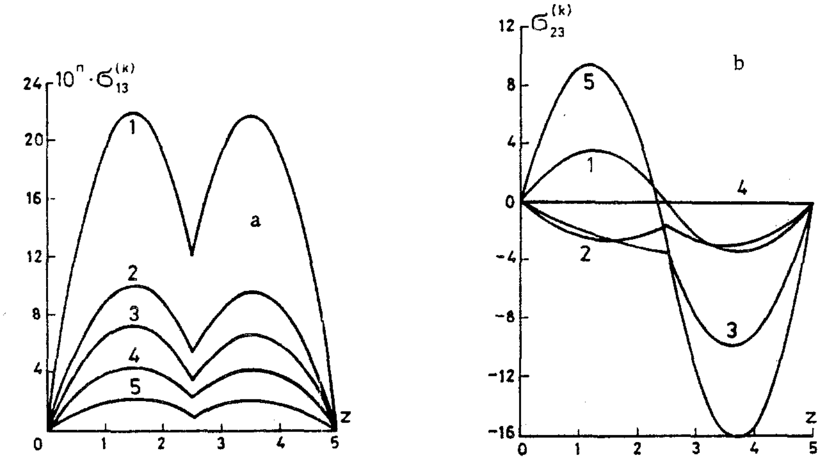

Fig. 1

is not considered, so the normal displacement (deflection) w is independent of the transverse coordinate z. Then the principle of virtual displacements was used to obtain 2N + 3 nonlinear equilibrium equations in the unknown functions u_i, w, and $\beta_i^{(k)}$. The overall order of the resolvent system of differential equations is 4N + 6. The equations of nonlinear transverse vibrations of shallow shells were derived later in [13].

Mathematical aspects of the resulting theory were examined in [14, 15], where proofs were given for several theorems on the properties of the root of the resolvent characteristic equation. The studies [16, 17] were devoted to special problems in the theory of multilayered plates. Cylindrical shells and beams made of composites were examined in [18-20].

The Grigolyuk–Chulkov theory was first generalized to nonshallow multilayered anisotropic shells in [11]. General relations for a three-dimensional continuum were used to obtain geometrically nonlinear equilibrium equations of multilayered shells written in arbitrary curvilinear coordinates. Here, the investigators considered the change in the metric over the thickness of the shell. A variant of the nonlinear theory of multilayered anisotropic shells was constructed in [21].

Unfortunately, the approaches examined above have one shortcoming. The transverse shear strains and (due to the use of Hooke's law) transverse shear stresses are distributed uniformly over the thickness of the k-th layer. In [22, 23], the authors used independent kinematic (1.1) and static hypotheses

$$\sigma_{i3}^{(k)}=f_0(z)\mu_i^{(0)}+f_k(z)\mu_i^{(k)}+p_i^-+z(p_i^+-p_i^-)/h \qquad (1.2)$$

to construct a consistent (from the viewpoint of the mixed variational principle) geometrically nonlinear variant of the theory of nonshallow multilayered anisotropic shells in which the transverse shear stresses are continuous functions of the transverse coordinate everywhere in the shell – including at the interfaces between the layers – and take prescribed values on the boundary surfaces. In (1.2), $f_0(z)$ and $f_k(z)$ are a priori assigned functions. These functions are continuous and satisfy the conditions $f_0(0)=f_0(h)=0$, $f_k(z)\equiv 0$, $z\in[0,\delta_{k-1}]\cup[\delta_k,h]$; $p_i^-=\sigma_{i3}^{(1)}(0)$, $p_i^+=\sigma_{i3}^{(N)}(h)$ are the shearing loads acting on the plane surfaces of the shell. No special restrictions were imposed on the form of the functions $f_0(z)$, $f_k(z)$ in constructing the theory, and the fact that they are quadratic parabolas is considered only in the solution of specific problems.

The authors of [22, 23] obtained 2N + 3 equilibrium equations in unit forces and moments and the corresponding boundary conditions. They also derived N + 1 integral elastic relations to express the relationship between the transverse shear strains and the transverse shear stresses. Such relations are important in the variant of multilayered shell theory being discussed, since they make it possible to express the "excess" functions $\mu_i^{(0)}$, $\mu_i^{(k)}$, from (1.2) through the kinematic shell characteristics u_i, w, $\beta_i^{(k)}$.

We will use a simple example to demonstrate the effectiveness of independent kinematic and static hypotheses (1.1), (1.2). Let a two-layered cross-reinforced cylindrical shell with fixed ends be subjected to tension in the axial direction. The article [24] gave the characteristics of a shell with layers made of a boron–epoxy composite. Figure 1 shows the distribution of the transverse shear stresses $\sigma_{13}^{(k)}$ over the thickness of the packet (h = 5 mm) in the shell section located 10 mm from the end (L = 100) as a function of the angle of reinforcement $\gamma = 0°, 15°, 45°, 60°, 75°$ (see curves 4, 1, 5, 3, 2; n = 0, 0, –1, 0, 0). It can be seen that the stresses $\sigma_{13}^{(k)}$ are distributed over the packet thickness in accordance with a law which is close to parabolic. As regards the stresses $\sigma_{23}^{(k)}$, they generally have a nonparabolic distribution. This case is postulated in most of the refined theories of multilayered shells with which we are familiar. Here, the $\sigma_{13}^{(k)}$ and $\sigma_{23}^{(k)}$ are of the same order, which is indicative of the substantial effect of anisotropy on the stress–strain state of the structure. An analysis was also made of edge effects in cross-reinforced cylindrical shells. It turned out that the character of the change in the curves of transverse shear stress is abruptly altered near a free edge. Thus, the stresses $\sigma_{13}^{(k)}$ are distributed in accordance with a sinusoidal law, while the stresses $\sigma_{23}^{(k)}$ are distributed according to a law which is close to parabolic.

A more general variant of the nonlinear theory of multilayered shells was developed in [25] on the basis of interdependent kinematic and static hypotheses [see (1.2)].

The tangential displacements are distributed over the thickness of the k-th layer in accordance with a nonlinear law, since they are determined from the formulas for the transverse shear strains of the k-th layer:

$$\varepsilon_{i3}^{(k)} = (G_{i3}^{(k)})^{-1}[f_0(z)\mu_i^{(0)} + f_k(z)\mu_i^{(k)} + p_i^- + z(p_i^+ - p_i^-)/h]$$

by integrating them over the transverse coordinate z with allowance for the continuity conditions for the tangential displacements:

$$u_i^{(n)}(\delta_n) = u_i^{(n+1)}(\delta_n) \quad (n = 1, 2, \ldots, N-1). \tag{1.3}$$

Here, $G_{i3}^{(k)}$ are the transverse shear moduli of the k-th layer. Given this approach, the overall order of the resolvent system of differential equations obtained from the principle of virtual displacements increases by two and becomes equal to 4N + 8. The order is 12 for a one-layer shell (N = 1).

As is known, in the sphere of computer technology, researchers have yet to solve the problem of storing large files of data in the computer memory. Thus, the broken-line hypothesis has received more and more attention among scientists. The broken-line hypothesis (1.1) was used in [26] to calculate laminated rods of nonsymmetrical structure. The neutral line of the rod was chosen as the initial line, which complicates the subsequent analysis somewhat. The resolvent system of N + 1 ordinary differential equations, of the order 2(N + 1), in the angles of rotation of the normal $\beta_1^{(k)}$ and the deflection w is solved by the finite-difference method. The authors of [27, 28] devised an algorithm for numerical solution of problems of the strength of multilayered orthotopic shells in a geometrically nonlinear formulation. In the case of axisymmetric deformation of a closed shell of revolution, a system of 2(N + 2) ordinary differential equations written in normal form is solved. A more general algorithm was developed in [23, 29], where a normal system of ordinary differential equations of the order 4N + 6 was used to determine the stress–strain state of an axisymmetrically loaded and fastened shell of revolution made of anisotropic (nonorthotopic) materials. The solution vector in this case has the form

$$\mathbf{Y} = [T_{11}, N_1, \Phi_{11}^{(1)}, \ldots, \Phi_{11}^{(N)}, T_{12}, \Phi_{12}^{(1)}, \ldots, \Phi_{12}^{(N)},$$
$$u_1, w, \beta_1^{(1)}, \ldots, \beta_1^{(N)}, u_2, \beta_2^{(1)}, \ldots, \beta_2^{(N)}]^T,$$

where T_{ij} and Q_i are the unit tangential and transverse forces; N_i are generalized (in the Fepple–Karman sense) transverse forces; $\Phi_{ij}^{(k)}$ are the generalized unit forces of the k-th layer; i, j = 1, 2. The authors of [29] solved the practically important problem of the stress state of a modern pneumatic tire loaded at the service pressure. The geometric parameters of the initial (internal) surface of the tire were determined by means of a smoothing spline-approximation first proposed in [30]. By the latter, we mean shells in which the meridian is given on a plane by a discrete set of points whose coordinates contain random measurement errors.

The wide use of computers in engineering calculations has led certain investigators to prefer the use of the broken-line hypothesis in a form different from (1.1). The essence of the hypothesis remains the same with these modifications, but its numerical realization on a computer is somewhat easier. The authors of [31, 32] proposed as the unknown function the tangential displacements of the outer surfaces of the shell $v_i^{(0)}=u_i$, $v_i^{(N)}$ and the interfaces of the layers $v_i^{(n)}$ $(n=1, 2, \ldots, N-1)$ as well as the deflection w. In [31], the deflection was independent of the transverse coordinate z. The broken-line hypothesis can be written as follows in the new functions

$$u_i^{(k)}=u_{i0}^{(k)}+z_k u_{i1}^{(k)} \quad (-h_k/2\leqslant z_k\leqslant h_k/2); \tag{1.4}$$

$$u_{i0}^{(k)}=\frac{v_i^{(k)}+v_i^{(k-1)}}{2}; \quad u_{i1}^{(k)}=\frac{v_i^{(k)}-v_i^{(k-1)}}{h_k}, \tag{1.5}$$

where $u_{i0}^{(k)}$, $u_{i1}^{(k)}$ are the tangential displacements and angles of rotation of the normal to the middle surface of the k-th layer; z_k is the local transverse coordinate, reckoned from the middle surface of the k-th layer in the direction of extension of the normal toward the initial surface of the shell. The study [31] solved a complex dynamic problem involving calculation of an axisymmetrically loaded and fastened multilayered orthotopic shell in the case of elastoplastic deformation of the layers. Later, the broken-line hypothesis was used in the form (1.4)-(1.5) to study vibrations of multilayered orthotopic plates and cylindrical shells having layers made of elastic and viscoelastic materials [33-35].

In [36], the broken-line hypothesis was used in yet another form to analyze multilayered anisotropic plates. This form of the hypothesis was first proposed by Gotteland in [37]:

$$u_i^{(k)}=v_i^{(k-1)}+(z-\delta_{k-1})\frac{v_i^{(k)}-v_i^{(k-1)}}{h_k}; \quad u_3^{(k)}=w. \tag{1.6}$$

Transformation from the local coordinates α_1, α_2, z_k in (1.4) to the global coordinates α_1, α_2, z shows that the two systems of kinematic hypotheses (1.4), (1.5), and (1.6) are equivalent.

The Timoshenko kinematic hypothesis was used in the form (1.4) in [38] for each orthotopic layer to describe the dynamic behavior of plates. If continuity conditions (1.3) are then satisfied, we arrive at the broken-line hypothesis (1.1). The only difference in the latter here is that instead of the tangential displacements of the "lower" plane surface of the plate u_i in [38] the tangential displacements of the middle surface of the "bottom" layer $u_{i0}^{(1)}$ are used. The Hamilton—Ostrogradskii principle was used to obtain 2N + 3 equations of motion in the functions $u_{i0}^{(1)}$, w, $u_{i1}^{(k)}$. A comparison was also made with two theories obtained on the basis of hypotheses applicable for the packet as a whole.

In [39], these equations were generalized to the case of allowance for interlayer shear stresses $\tau_i^{(n)}=\sigma_{i3}^{(n)}(\delta_n)=\sigma_{i3}^{(n+1)}(\delta_n)$ (n = 1, 2, ..., N − 1). Together with the functions $u_{i0}^{(1)}$, w, $u_{i1}^{(k)}$, the interlayer shear stresses $\tau_i^{(n)}$ were included in the equilibrium equations obtained in [39] by the solution of a variational problem for a conditional extremum (the functions $\tau_i^{(n)}$ were used as Lagrangian multipliers). Let us turn our attention to the fact that $\tau_i^{(0)}=p_i^-$ and $\tau_i^{(N)}=p_i^+$ are known functions. The fact that additional unknown functions $\tau_i^{(n)}$ are present in the refined equations 2(N − 1) undoubtedly complicates the problem. However, in a first approximation, it also makes it possible to solve one of the most important problems in the mechanics of composite materials. If we exclude $\tau_i^{(n)}$ from the resolvent equations obtained in [33], we arrive at the equations in [7].

This approach later received wide support and was used by many investigators — such as in [40] — to calculate interlayer shear stresses. The essence of the approach here is that each anisotropic layer of a shell can be described by equations of a Timoshenko-type shell theory. Thus, for the shell as a whole, we have 5N equilibrium equations in the function: $u_{i0}^{(k)}$, $u_{i1}^{(k)}$, w [see (1.4)] and $\tau_i^{(n)}$, $\tau_3^{(n)}$, where $\tau_3^{(n)}=\sigma_{33}^{(n)}(\delta_n)=\sigma_{33}^{(n+1)}(\delta_n)$ are the interlayer normal stresses (n = 1, 2, ..., N − 1). It is not hard to see that here we introduced 4N + 1 sought kinematic and 3(N − 1) static functions. As the auxiliary equation introduced to close the problem, we employ 2(N − 1) continuity equations (1.3) for the tangential components of the displacement vector. On the other hand, if we immediately use (1.3), then the number of kinematic functions subject to determination can be reduced from 4N + 1 to 2N + 3. In fact, this means that the broken-line hypothesis is used in the form (1.1) for the entire packet.

Unfortunately, the method described in [40, 41] was realized only for a two-layer cylindrical shell. A general approach to the solution of the problems being discussed was given in [42-44], where a productive method was developed for solving symmetrical problems of the theory of plates and shells made of composite materials. Having excluded the displacements from the resolvent system of equations, the authors of [43] arrived at a system of 2(N − 1) Volterra equations in the interlayer shear stresses and normal stresses $\tau_1^{(n)}$, $\tau_3^{(n)}$ $(n=1, 2, \ldots, N-1)$. This system is solved with the aid of the Laplace transform. The authors of [45] examined the axisymmetric deformation of multilayered cylindrical shells in the case of an imperfect adhesive joint between the layers.

An original approach to the construction of a theory of multilayered orthotopic shells was proposed in [46, 47]. The following approximation was adopted for the transverse shear stresses of the k-th layer

$$\sigma_{i3}^{(k)}=f_i^{(k)}(z_k)\varphi_i^{(k)}+z_k/h_k(\tau_i^{(k)}-\tau_i^{(k-1)})+1/2(\tau_i^{(k)}+\tau_i^{(k-1)}), \tag{1.7}$$

where $f_i^{(k)}(z_k)$ are a priori assigned functions continuous on the interval $[-h_k/2, h_k/2]$ and satisfying the conditions $f_i^{(k)}(\pm h_k/2)=0$; $\varphi_i^{(k)}$ are the sought shear functions. The transverse contraction of the shell over the thickness is not considered. It can be seen from Eq. (1.7) that the transverse shear stresses are continuous functions of the transverse coordinate, while on the plane surfaces they satisfy the assigned boundary conditions $\sigma_{i3}^{(1)}(-h_1/2)=\tau_i^{(0)}$; $\sigma_{i3}^{(N)}(h_N/2)=\tau_i^{(N)}$. Subsequent introduction of $\sigma_{i3}^{(k)}$ from (1.7) into the Hooke's law relations for the transverse components of the stress and strain tensors leads to formulas for the transverse shears of the k-th layer. Integrating the resulting relations over the thickness of the k-th layer and using continuity conditions (1.3), we obtain the final formulas for the tangential displacements $u_i^{(k)}$. The equations of motion of multilayered orthotropic cylindrical shells were statically obtained. One of the shortcomings of the theory in [46] is that variational principles were not used to establish the connection between the kinematic and force characteristics of the shell. The question of internal forces and moments which are compatible with the chosen hypotheses is of fundamental importance for any refined theory, since failure to allow for their effect often leads to a qualitatively different problem described by a lower-order system of differential equations.

A similar approach was described in [48], where contraction of a plate over its thickness was also not considered, i.e. $u_3^{(k)} = w$. The remaining characteristics of the stress–strain state of the plate were represented in the form of series in Legendre polynomials. The use of such representations in reducing a three-dimensional problem of the theory of elasticity to a two-dimensional problem of the theory of multilayered plates and shells leads to an infinite system of differential equations. Thus, two terms are retained in the series for the tangential components of the displacement vector and stress and strain tensors, while three terms are kept in the series for the transverse shear stresses. If we now return from mathematics to mechanics, we can see that the layers of the plate are actually deformed in accordance with the Timoshenko shear model. However, this approach is significantly different from that described in [42-44], which makes it possible to derive analytic formulas for the transverse shear stresses and follow the law of their change over the thickness of the packet. In fact, after satisfaction of the continuity conditions at the interfaces and the boundary conditions on the faces of the plate, the author of [48] obtained the formulas

$$\sigma_{i3}^{(k)}=\left[P_0\left(\frac{2z_k}{h_k}\right)-P_2\left(\frac{2z_k}{h_k}\right)\right]\frac{Q_i^{(k)}}{h_k}+\frac{1}{2}P_1\left(\frac{2z_k}{h_k}\right)(\tau_i^{(k)}-\tau_i^{(k-1)})+\frac{1}{2}P_2\left(\frac{2z_k}{h_k}\right)(\tau_i^{(k)}+\tau_i^{(k-1)}). \tag{1.8}$$

Here, $P_0(\zeta)=1$; $P_1(\zeta)=\zeta$; $P_2(\zeta)=(3\zeta^2-1)/2$ are Legendre polynomials; $Q_i^{(k)}$ are the unit transverse forces of the k-th layer. Comparison of approximations (1.7) and (1.8) shows that in the case of a parabolic dependence of the functions $f_i^{(k)}$ on z_k, the formulas are the same from a practical point of view. Thus, by using the representations in [48], it is possible to offer an acceptable mechanical interpretation of the Hsu–Wang hypothesis.

2. None of the studies analyzed in Part 1 considered transverse contraction of the layers. Thus, here we will concentrate exclusively on this question. Within the framework of the Grigolyuk–Chukov theory of multilayered shells [49], allowance was made in [50, 51] for the normal strains in the transverse direction. The authors examined a multilayered plate made of an arbitrary number of transversely isotropic layers. The tangential and normal displacements in the k-th layer of the plate are approximated by linear functions of the transverse coordinate (see (1.1) with i = 1, 2, 3). The principle of virtual displacements was used to derive 3(N + 1) equilibrium equations in the unit forces and moments. The

overall order of the resolvent system of differential equations was $6(N + 1)$.

The studies [10, 52-55] then examined multilayered anisotropic shells in both geometrically linear and geometrically nonlinear formulations in arbitrary curvilinear coordinates. These investigations also examined multilayered isotropic shells in a physically nonlinear formulation. It was assumed that all of the components of the displacement vector are distributed over the thickness of the k-th layer in accordance with the Timoshenko kinematic hypothesis (see (1.4) with $i = 1, 2, 3$). The mixed variational principle was used to obtain shell equilibrium equations, the corresponding boundary conditions, and the strain continuity equations. The uniqueness theorem, Clapeyron's theorem, Betti's reciprocity theorem, the principle of the potential strain energy minimum, and the principle of the minimum of additional work were proven in the linear theory of multilayered shells. Investigators also demonstrated the self-adjoint nature of the linear boundary-value problem and the orthogonality of the modes of natural vibration. In an even more general formulation, nonlinear equations of multilayer anisotropic shells were constructed in [12, 56] with allowance for large strains.

The study [57] presented an improved (compared to [21]) variant of the geometrically nonlinear theory of multilayered anisotropic shells with layers of variable thickness. The authors of [37, 58] generalized the equations of motion of multilayered anisotropic plates and shallow shells obtained in [13] to the case of allowance for all components of the stress–strain state and the inertia of the structure.

Let us take a more detailed look at the methods of calculation used for multilayered anisotropic shells of complex geometry. The study [59] examined a modified variant of the Grigolyuk–Chulkov theory. The introduction of a corrected Poisson's ratio for each shell layer made it possible to write the resolvent equations in compact form, which in turn simplified the numerical realization of axisymmetric problem, of the statics and thermoelasticty of multilayered shells of revolution with an arbitrarily shaped meridian on a computer [60]. The authors of [32] constructed a variant of the theory of multilayered anisotropic shells in central bending (after K. Z. Galimov). As the sought functions, the investigators took the displacements of the outer surfaces of the shell $v_i^{(0)}$, $v_i^{(N)}$ and the interfaces of the layers $v_i^{(n)}$ ($n = 1, 2, \ldots, N - 1$; also see (1.4), (1.5) with $i = 1, 2, 3$), which leads to a three-diagonal band matrix of differential operators for the problem. An examination was made of shells of complex geometry with layers of constant [32] and variable [61] thickness. A practically important model of a multilayered shell was proposed in [62]. In this model, the shell is composed of stiff moment layers of variable thickness and thin momentless layers.

A variant of the theory of multilayered anisotropic shells of revolution of variable thickness which considers kinematic and physical nonlinearity was constructed in [63, 64]. The theory is based on adoption of Timoshenko-type kinematic hypotheses in each layer. An algorithm was developed for numerical solution of axisymmetric problems of shells of revolution of complex form on a computer.

A nonlinear variant of the theory of multilayered shells which makes it possible to study the actual distribution law for transverse shear stresses over the thickness of the packet [23, 29] was later generalized in [65] to the case of local contraction of the layers over the thickness. The following approximation was adopted for the transverse normal stresses

$$\sigma_{33}^{(k)} = g_0(z)\psi^{(0)} + g_k(z)\psi^{(k)} + q^- + z(q^+ - q^-)/h. \qquad (2.1)$$

Here, q^- and q^+ are the normal loads acting on the faces of the shell; $g_0(z)$ and $g_k(z)$ are a priori assigned functions which are continuous and satisfy the conditions $g_0(0) = g_0(h) = 0$; $g_k(z) \equiv 0$, $z \in [0, \delta_{k-1}] \cup [\delta_k, h]$. Whereas the functions $f_0(z)$, $f_k(z)$ from (1.2) change in accordance with the quadratic parabola law, it was recommended in [65] that cubic polynomials be taken as the functions $g_0(z)$, $g_k(z)$. Such a choice is consistent with the third equilibrium equation of the theory of elasticity. The distribution of the transverse components of the stress tensor (1.2), (2.1) over the shell thickness was subjected to a parametric analysis. The results obtained here have application to the design of modern pneumatic tires.

The first use of Legendre polynomials to construct the theory of multilayered shells was evidently made in [66, 67]. The authors here examined a thick shell of anisotropic layers. The shell was compressible in the transverse direction. The displacements within each layer were represented in the form of series in Legendre polynomials. The principle of virtual displacements was used to obtain geometrically nonlinear equations of equilibrium of the shell in the unit forces and moments. The corresponding boundary conditions were also obtained. A detailed study was made of the errors introduced by the number of terms retained in the series for the expansion of the sought functions. Numerical results illustrating the effectiveness of the proposed approach were given in [68, 69].

The investigations [70-72] presented all of the components of the displacement vector and the stress and strain tensors of the k-th layer of a shell in the form of series in Legendre polynomials. Using the general theorem on the approximation of a function and its first derivative by the authors proposed retaining m + 1 terms in the series for the tangential components of the stress tensor and n + 3 and n + 2 terms ($m \leq n + 1$) in the series for the transverse shear and normal stresses. These expansions mean that m + 1 and n + 1 terms should be kept in the series for the tangential and normal displacements, respectively. This approximation of the problem is called the {m, n}-approximation in the theory of laminated anisotropic shells. The monograph [70] focused mainly on the {1, 0}-approximation (which was analyzed in fairly great detail in Part 1), in which transverse contraction of the layers is not considered [48].

Included under the topic of our survey is the theory in [73], in which the order of the resolvent system of differential equations depends on the number of layers. Kinematic hypotheses are not used in explicit form in this theory, and all of the constructions are based on the concept of a mixture. This concept was developed earlier in the mechanics of composite materials.

We should also note, the approach taken in [74], which is based on numerical integration of linear equations of the theory of elasticity for a multilayered anisotropic shell. With particular types of boundary conditions, the numerical results obtained on the basis of this approach are nearly exact and can be used as standard values.

3. An analysis of the studies devoted to developing general models of elastic multilayered shells with allowance for local effects showed that the efforts of most investigators have been concentrated on the construction of models in which the displacement vector and the transverse components of the stress tensor are continuous functions of the transverse coordinate over the entire shell – including the interfaces – and they satisfy assigned boundary conditions on the external surfaces. Also, the equations of Hooke's law for the transverse components of the stress tensor should be satisfied exactly, rather than in an integral sense (as in a Timoshenko-type shell theory).

These conditions and equations can be satisfied if, for example, we proceed in the following manner. Let the components of the displacement vector be represented in the form of power series. Having kept the first four terms in the series, we obtain

$$u_j^{(k)} = u_{j0}^{(k)} + \zeta_k u_{j1}^{(k)} + \zeta_k^2 u_{j2}^{(k)} + \zeta_k^3 u_{j3}^{(k)}, \tag{3.1}$$

where $\zeta_k = 2z_k/h_k;\ -h_k/2 \leqslant z_k \leqslant h_k/2;\ j=1,\ 2,\ 3;\ k=1, 2, \ldots, N$.

The more convenient formulas below follow from Eqs. (3.1) and the continuity conditions at the interfaces

$$u_j^{(k)} = \frac{v_j^{(k)} + v_j^{(k-1)}}{2} + \zeta_k \frac{v_j^{(k)} - v_j^{(k-1)}}{2} + (\zeta_k^2 - 1)\, u_{j2}^{(k)} + (\zeta_k^3 - \zeta_k)\, u_{j3}^{(k)}. \tag{3.2}$$

Here, $v_j^{(n)}$ $(n=0, 1, \ldots, N)$ are the tangential and normal displacements of the faces of the layers.

We will use Hooke's law to exclude the remaining functions $u_{j2}^{(k)}$, $u_{j3}^{(k)}$. For a linearly elastic multilayered isotropic plate, Hooke's law can be written in the form

$$\begin{gathered}\sigma_{i3}^{(k)} = G_k \left(\frac{\partial u_i^{(k)}}{\partial z_k} + \frac{\partial u_3^{(k)}}{\partial x_i} \right) \quad (i=1, 2);\\ \sigma_{33}^{(k)} = \lambda_k \left(\frac{\partial u_1^{(k)}}{\partial x_1} + \frac{\partial u_2^{(k)}}{\partial x_2} \right) + (\lambda_k + 2G_k) \frac{\partial u_3^{(k)}}{\partial z_k},\end{gathered} \tag{3.3}$$

where x_1 and x_2 are Cartesian coordinates of the plane of reference; λ_k and G_k are elastic constants of the k-th layer. Inserting the displacements (3.2) into (3.3) and using the continuity conditions $\sigma_{j3}^{(k)}(h_k/2) = \tau_j^{(k)}$; $\sigma_{j3}^{(k)}(-h_k/2) = \tau_j^{(k-1)} (j=1, 2, 3)$, we obtain the expressions

$$u_{i2}^{(k)}=\frac{h_k}{8}\left(\frac{\tau_i^{(k)}-\tau_i^{(k-1)}}{G_k}-\frac{\partial(v_3^{(k)}-v_3^{(k-1)})}{\partial x_i}\right);$$

$$u_{i3}^{(k)}=\frac{h_k}{8}\left(\frac{\tau_i^{(k)}+\tau_i^{(k-1)}}{G_k}-2\frac{v_i^{(k)}-v_i^{(k-1)}}{h_k}-\frac{\partial(v_3^{(k)}+v_3^{(k-1)})}{\partial x_i}\right)\quad(i=1,2);$$

$$u_{32}^{(k)}=$$

$$=\frac{h_k}{8}\left[\frac{\tau_3^{(k)}-\tau_3^{(k-1)}}{\lambda_k+2G_k}-\frac{\lambda_k}{\lambda_k+2G_k}\left(\frac{\partial(v_1^{(k)}-v_1^{(k-1)})}{\partial x_1}+\frac{\partial(v_2^{(k)}-v_2^{(k-1)})}{\partial x_2}\right)\right]; \tag{3.4}$$

$$u_{33}^{(k)}=\frac{h_k}{8}\left[\frac{\tau_3^{(k)}+\tau_3^{(k-1)}}{\lambda_k+2G_k}-2\frac{v_3^{(k)}-v_3^{(k-1)}}{h_k}-\frac{\lambda_k}{\lambda_k+2G_k}\times\right.$$

$$\left.\times\left(\frac{\partial(v_1^{(k)}+v_1^{(k-1)})}{\partial x_1}+\frac{\partial(v_2^{(k)}+v_2^{(k-1)})}{\partial x_2}\right)\right].$$

If the surface loads are assigned on the external surfaces $z_1 = -h_1/2$, $z_N = h_N/2$, then the functions $\tau_j^{(0)}$, $\tau_j^{(N)}$ are assumed to be known. Otherwise, the known quantities are the displacement of the boundary surfaces $u_j^{(0)}$, $u_j^{(N)}$. Thus, only 6N functions will be independent out of the 6(N + 1) functions which figure into Eqs. (3.2), (3.4).

Approximation of the displacement field in a multilayered plate by means of Eqs. (3.2), (3.4) satisfies all of the above-noted requirements and can serve as a basis for constructing internally consistent models of multilayered anisotropic shells.

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DETERMINATION OF THE LOAD-CARRYING CAPACITY OF COMPRESSED ELASTOPLASTIC SHELLS IN THE ABSENCE AND PRESENCE OF STRESS CONCENTRATORS AT ELEVATED TEMPERATURES

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A considerable number of studies – in particular, [1-8] – has been devoted to theoretical and experimental investigation of the load-carrying capacity and stability of composite shells. The study [8] discussed analytical methods, based on linear fracture mechanics, for determining the residual compressive strength of composite elements containing stress raisers in the form of through cracks and holes. Here, on the basis of an experimental study, we examine the applicability of methods of calculating the strength of compressed carbon-fiber-reinforced plastic shells in the presence and absence of stress concentrators at elevated temperatures (T = 393 K; 443 K). We also conducted tests at room temperature (T = 293 K) to obtain comparative data from tests of several monolithic cylindrical shells made of carbon-plastic KMU-1LM.

The 25 cylindrical shells of carbon-plastic KMU-1LM designated for testing, with a fiber arrangement of the type [0/90±45] were made by the filament-winding method on a five-coordinate machine with programmed control. Here, binder-impregnated carbon tape was wound about a metallic mandrel and subsequently formed in an autoclave. The ends of the shells were reinforced with a glass-cloth edge piping to prevent buckling during compressive loading. We varied the reinforcement scheme, the wall thickness (δ = 2.9-5.07 mm), and length (ℓ = 720 and 1325 mm). The shells had roughly the same diameter – 790 mm. Table 1 shows geometric parameters of the carbon-plastic shells along with the reinforcement schemes, tests results and results of calculations.

The shells were tested on a universal electromechanical machine. The error of the force-measuring system of the machine was no greater than ±1%. The loads were transmitted to the ends of the shells through slabs hinged to the stationary base of the machine and its movable top cross-piece. The shells were jacketed and centered relative to the loading axis prior to testing to ensure uniformity of application of the compressive loads about the

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