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A METHOD OF SOLVING THREE-DIMENSIONAL PROBLEMS OF ELASTICITY FOR LAMINATED COMPOSITE PLATES

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An efficient method of solving 3D elasticity problems for thick and thin laminated composite plates is presented. It is based on a new concept of reference surfaces inside the plate. According to this concept, into each nth layer, I_n arbitrary reference surfaces parallel to the midsurface are introduced, and the displacement vectors of these surfaces are chosen as unknown functions. Such a choice allows one to represent the governing equations of the high-order theory of plates proposed in a very compact form and to derive strain–displacement relationships correctly describing all rigid-body motions of laminated plates.

Introduction

As is known, the traditional way of constructing theories of plates and shells consists in expansion of displacements into power series in terms of the transverse coordinate θ_3, reckoned along the external normal to the midsurface [1-3]. For an approximate representation of the displacement field, finite segments of power series can be used, since the main purpose of the theory of plates and shells is to derive approximate solutions to the problems of three-dimensional elasticity theory. The idea of this approach goes back to studies by Cauchy [4]. A theory of shells based on the expansion of the displacement field into series of Legendre polynomials in terms of the transverse coordinate has also been developed [5]. However, the apparent advantage of this theory is lost when applied to problems on the statics of thick plates and shells where a great number of terms must be retained in the corresponding expansion so that to obtain admissible results.

An alternative concept is connected with introduction of I_n reference surfaces $\Omega^{(n)1}, \Omega^{(n)2},..., \Omega^{(n)I_n}$ into each layer, parallel to the shell midsurface, in order to use the displacement vectors $\mathbf{u}^{(n)1}, \mathbf{u}^{(n)2},..., \mathbf{u}^{(n)I_n}$ of these surfaces as sought-for

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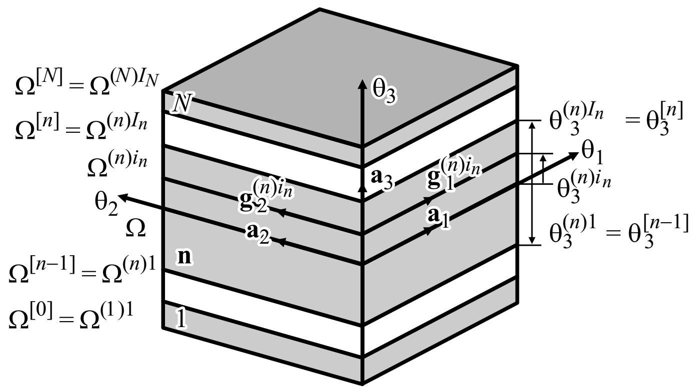

Fig. 1. Arrangement of reference surfaces in a plate.

functions. In this case, the surfaces $\Omega^{(n)1}$ and $\Omega^{(n)I_n}$ coincide with the interfaces of layers $\Omega^{[n-1]}$ and $\Omega^{[n]}$. Hereinafter, the superscript *n* indicates that the quantity belongs to an *n*th layer and takes values 1, 2, …, *N*, where *N* is the number of layers in the package. Such a choice of required functions, with subsequent use of the Lagrange polynomials of degree $I_n - 1$ in 3D approximations of displacements, allows one to construct deformation relations that exactly represent the displacements of the layered plate as a rigid body.

Kinematics of Plates and Deformation Relations

Let us consider a plate of constant thickness *h*. Its midsurface Ω is referred to curvilinear orthogonal coordinates θ_1 and θ_2, but the coordinate θ_3 is reckoned in the direction of normal. The base vectors of the midsurface are

$$\boldsymbol{a}_\alpha = \mathbf{r}_{,\alpha} = A_\alpha \mathbf{e}_\alpha, \quad \boldsymbol{a}_3 = \mathbf{e}_3, \tag{1}$$

where $\mathbf{r} = \mathbf{r}(\theta_1, \theta_2)$ is the radius-vector of a point on the midsurface; $A_\alpha(\theta_1, \theta_2)$ are coefficients of the first quadratic form; $\mathbf{e}_\alpha(\theta_1, \theta_2)$ are the unit vectors of tangents to the coordinate lines θ_α; $\mathbf{e}_3$ is the unit vector of normal to the midsurface. The base vectors of the reference surface $\Omega^{(n)i_n}$, located inside an *n*th layer (Fig. 1), have the form

$$\mathbf{g}_\alpha^{(n)i_n} = \mathbf{R}_{,\alpha}^{(n)i_n} = A_\alpha \mathbf{e}_\alpha, \quad \mathbf{g}_3^{(n)i_n} = \mathbf{e}_3, \tag{2}$$

where $\mathbf{R}^{(n)i_n} = \mathbf{r} + \theta_3^{(n)i_n}\mathbf{e}_3$ is the radius-vector of a point on the reference surface $\Omega^{(n)i_n}$, and $\theta_3^{(n)i_n}$ is the transverse coordinate of the surface $\Omega^{(n)i_n}$. In this case, the relations

$$\begin{gathered}\theta_3^{(1)1} = \theta_3^{[0]} = -h/2, \quad \theta_3^{(N)I_N} = \theta_3^{[N]} = h/2, \\ \theta_3^{(m)I_m} = \theta_3^{(m+1)1} = \theta_3^{[m]} \quad (m = 1, 2, ..., N-1).\end{gathered} \tag{3}$$

are valid. Hereinafter, $\alpha, \beta = 1, 2$; $i, j, k, m = 1, 2, 3$; $i_n, j_n, k_n = 1, 2, .., I_n$.

The base vectors of the reference surface $\Omega^{(n)i_n}$ in the deformed state (Fig. 2) are found from the formulas

$$\bar{\mathbf{g}}_\alpha^{(n)i_n} = \bar{\mathbf{R}}_{,\alpha}^{(n)i_n} = \mathbf{g}_\alpha^{(n)i_n} + \mathbf{u}_{,\alpha}^{(n)i_n}, \quad \bar{\mathbf{g}}_3^{(n)i_n} = \mathbf{e}_3 + \boldsymbol{\beta}^{(n)i_n}, \tag{4}$$

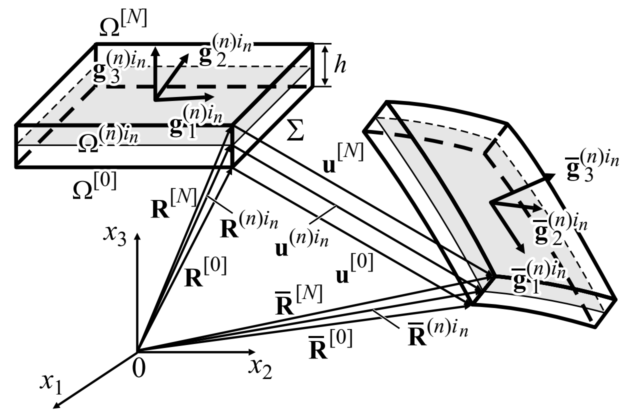

Fig. 2. Initial and deformed configurations of a plate.

$$\bar{\mathbf{R}}^{(n)i_n} = \mathbf{R}^{(n)i_n} + \mathbf{u}^{(n)i_n}, \tag{5}$$

$$\mathbf{u}^{(n)i_n} = \mathbf{u}(\theta_3^{(n)i_n}), \quad \boldsymbol{\beta}^{(n)i_n} = \mathbf{u}_{,3}(\theta_3^{(n)i_n}), \tag{6}$$

where $\bar{\mathbf{R}}^{(n)i_n}$ is the radius-vector of a point on the deformed surface $\Omega^{(n)i_n}$; $\mathbf{u}^{(n)i_n}(\theta_1,\theta_2)$ is the displacement vector of the reference surface $\Omega^{(n)i_n}$; $\boldsymbol{\beta}^{(n)i_n}(\theta_1,\theta_2)$ is the derivative of the displacement vector with respect to the transverse coordinate on the surface $\Omega^{(n)i_n}$. From the considerations of continuity of the displacement field, we have

$$\begin{gathered}\mathbf{u}^{(1)1} = \mathbf{u}^{[0]}, \quad \mathbf{u}^{(N)I_N} = \mathbf{u}^{[N]}, \\ \mathbf{u}^{(m)I_m} = \mathbf{u}^{(m+1)1} = \mathbf{u}^{[m]} \quad (m = 1, 2, ..., N-1),\end{gathered} \tag{7}$$

where $\mathbf{u}^{[m]}(\theta_1,\theta_2)$ are the displacement vectors of layer interfaces.

The components of strain tensor on the reference surface $\Omega^{(n)i_n}$, at $A_3 = 1$, have the form

$$2\varepsilon_{ij}^{(n)i_n} = 2\varepsilon_{ij}(\theta_3^{(n)i_n}) = \frac{1}{A_i A_j}(\bar{\mathbf{g}}_i^{(n)i_n} \cdot \bar{\mathbf{g}}_j^{(n)i_n} - \mathbf{g}_i^{(n)i_n} \cdot \mathbf{g}_j^{(n)i_n}). \tag{8}$$

Introducing base vectors (2) and (4) into the deformation relations of three-dimensional elasticity theory (8) and neglecting the nonlinear terms, we obtain

$$\begin{gathered}2\varepsilon_{\alpha\beta}^{(n)i_n} = \frac{1}{A_\alpha}\mathbf{u}_{,\alpha}^{(n)i_n} \cdot \mathbf{e}_\beta + \frac{1}{A_\beta}\mathbf{u}_{,\beta}^{(n)i_n} \cdot \mathbf{e}_\alpha, \\ 2\varepsilon_{\alpha 3}^{(n)i_n} = \boldsymbol{\beta}^{(n)i_n} \cdot \mathbf{e}_\alpha + \frac{1}{A_\alpha}\mathbf{u}_{,\alpha}^{(n)i_n} \cdot \mathbf{e}_3, \quad \varepsilon_{33}^{(n)i_n} = \boldsymbol{\beta}^{(n)i_n} \cdot \mathbf{e}_3.\end{gathered} \tag{9}$$

Let us present the vectors $\mathbf{u}^{(n)i_n}$ and $\boldsymbol{\beta}^{(n)i_n}$ in the orthonormal basis $\mathbf{e}_i$ according to the formulas

$$\mathbf{u}^{(n)i_n} = \sum_i u_i^{(n)i_n} \mathbf{e}_i, \tag{10}$$

$$\boldsymbol{\beta}^{(n)i_n} = \sum_i \beta_i^{(n)i_n} \mathbf{e}_i. \tag{11}$$

From expansion (10), with account of the known formulas of differentiation of base vectors $\mathbf{e}_i$ with respect to the coordinates θ_α, it follows that

$$\frac{1}{A_\alpha}\mathbf{u}_{,\alpha}^{(n)i_n} = \sum_i \lambda_{i\alpha}^{(n)i_n}\mathbf{e}_i, \tag{12}$$

where

$$\lambda_{\alpha\alpha}^{(n)i_n} = \frac{1}{A_\alpha}u_{\alpha,\alpha}^{(n)i_n} + B_\alpha u_\beta^{(n)i_n}, \quad \lambda_{\beta\alpha}^{(n)i_n} = \frac{1}{A_\alpha}u_{\beta,\alpha}^{(n)i_n} - B_\alpha u_\alpha^{(n)i_n} \quad (\beta \neq \alpha), \tag{13}$$

$$\lambda_{3\alpha}^{(n)i_n} = \frac{1}{A_\alpha}u_{3,\alpha}^{(n)i_n}, \qquad B_\alpha = \frac{1}{A_\alpha A_\beta}A_{\alpha,\beta} \quad (\beta \neq \alpha).$$

Introducing expansions (11) and (12) into Eqs. (9), we come to the scalar form of deformation relations

$$2\varepsilon_{\alpha\beta}^{(n)i_n} = \lambda_{\alpha\beta}^{(n)i_n} + \lambda_{\beta\alpha}^{(n)i_n}, \tag{14}$$

$$2\varepsilon_{\alpha 3}^{(n)i_n} = \beta_\alpha^{(n)i_n} + \lambda_{3\alpha}^{(n)i_n}, \qquad \varepsilon_{33}^{(n)i_n} = \beta_3^{(n)i_n}.$$

3D Approximations of the Fields of Displacements and Strains

We should note that, up to this point, no assumptions on the character of distribution of displacement and strain fields inside the plate have been made. Let the displacements be distributed in the transverse direction of the plate according to the following law:

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n} u_i^{(n)i_n}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{15}$$

where $L^{(n)i_n}\left(\theta_3\right)$ are the Lagrange polynomials of degree $I_n - 1$, defined by the formula

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{\theta_3 - \theta_3^{(n)j_n}}{\theta_3^{(n)i_n} - \theta_3^{(n)j_n}}. \tag{16}$$

In this case, $L^{(n)i_n}\left(\theta_3^{(n)j_n}\right) = 1$ at $j_n = i_n$ and $L^{(n)i_n}\left(\theta_3^{(n)j_n}\right) = 0$ at $j_n \neq i_n$.
Relations (6), (11), and (15) yield

$$\beta_i^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n})u_i^{(n)j_n}, \tag{17}$$

where $M^{(n)j_n} = L_{,3}^{(n)j_n}$ are polynomials of degree $I_n - 2$; according to Eq. (16), their values on the reference surface $\Omega^{(n)i_n}$ are found by the formulas

$$M^{(n)j_n}\left(\theta_3^{(n)i_n}\right) = \frac{1}{\theta_3^{(n)j_n} - \theta_3^{(n)i_n}} \prod_{k_n \neq i_n, j_n} \frac{\theta_3^{(n)i_n} - \theta_3^{(n)k_n}}{\theta_3^{(n)j_n} - \theta_3^{(n)k_n}} \quad (j_n \neq i_n), \tag{18}$$

$$M^{(n)i_n}\left(\theta_3^{(n)i_n}\right) = -\sum_{j_n \neq i_n} M^{(n)j_n}\left(\theta_3^{(n)i_n}\right).$$

Thus, the governing functions of the theory of plates suggested, $\beta_i^{(n)i_n}$, are represented as a linear combination of displacements of the reference surfaces $u_i^{(n)j_n}$.

The following step consists in the choice of a law of strain distribution across the thickness of the plate. It is obvious that the distribution of strains in the transverse direction must be coordinated with the distribution of displacements (15), i.e., we have

$$\varepsilon_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} \varepsilon_{ij}^{(n)i_n}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}. \tag{19}$$

Statement. Deformation relations (9) and (19), along with 3D approximations (15) and (17), exactly represent the displacement of the layered plate as a rigid body.

▹ Displacement of the reference surface $\Omega^{(n)i_n}$ as a rigid body can be presented in the form [6, 7]

$$(\mathbf{u}^{(n)i_n})^{\text{Rigid}} = \mathbf{\Delta} + \mathbf{\Phi} \times \mathbf{R}^{n(i_n)}, \tag{20}$$
$$\mathbf{\Delta} = \sum_i \Delta_i \mathbf{e}_i, \quad \mathbf{\Phi} = \sum_i \Phi_i \mathbf{e}_i,$$

where $\mathbf{\Delta}$ is the vector of translational displacement of the plate, and $\mathbf{\Phi}$ is the vector of rotations. From relations (20), Eqs. (2), and

$$\mathbf{\Delta}_{,\alpha} = \mathbf{0}, \quad \mathbf{\Phi}_{,\alpha} = \mathbf{0}, \tag{21}$$

it follows that

$$(\mathbf{u}_{,\alpha}^{(n)i_n})^{\text{Rigid}} = A_\alpha \mathbf{\Phi} \times \mathbf{e}_\alpha. \tag{22}$$

In view of the identities

$$\sum_{j_n} M^{(n)j_n}(\theta_3) = 0, \qquad \sum_{j_n} \theta_3^{(n)j_n} M^{(n)j_n}(\theta_3) = 1, \tag{23}$$

which in turn follow from the obvious identities

$$\sum_{j_n} L^{(n)j_n}(\theta_3) = 1, \qquad \sum_{j_n} \theta_3^{(n)j_n} L^{(n)j_n}(\theta_3) = \theta_3, \tag{24}$$

we find

$$(\boldsymbol{\beta}^{(n)i_n})^{\text{Rigid}} = \sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n})(\mathbf{u}^{(n)j_n})^{\text{Rigid}} = \mathbf{\Phi} \times \mathbf{e}_3. \tag{25}$$

Introducing Eqs. (22) and (25) into deformation relations (9), we have

$$2(\varepsilon_{ij}^{(n)i_n})^{\text{Rigid}} = (\mathbf{\Phi} \times \mathbf{e}_i)\mathbf{e}_j + (\mathbf{\Phi} \times \mathbf{e}_j)\mathbf{e}_i = 0, \tag{26}$$

which was to be proved. ◃

Variational Formulation of the Problem

Inserting strains (19) into the principle of virtual work and introducing the stress resultants

$$H_{ij}^{(n)i_n} = \int_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} \sigma_{ij}^{(n)} L^{(n)i_n} d\theta_3, \tag{27}$$

we come to the variational equation

$$\iint_\Omega \left[\sum_n \sum_{i_n} \sum_{i,j} H_{ij}^{(n)i_n} \delta\varepsilon_{ij}^{(n)i_n} - \sum_i \left(p_i^+ \delta u_i^{[N]} - p_i^- \delta u_i^{[0]} \right) \right] A_1 A_2 d\theta_1 d\theta_2 = \delta W_\Sigma, \tag{28}$$

where p_i^- and p_i^+ are the surface loads acting on the inner and outer surfaces of the plate, and W_Σ is the work of external forces operating on the lateral surface Σ.

We will limit ourselves to the consideration of linearly elastic materials, to which the relations of the generalized Hooke's law

$$\sigma_{ij}^{(n)} = \sum_{k,\ell} C_{ijk\ell}^{(n)} \varepsilon_{k\ell}^{(n)}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{29}$$

can be applied.

Now, introducing stresses (29) into Eq. (27) and taking into account the designation

$$D_{ijk\ell}^{(n)i_n j_n} = C_{ijk\ell}^{(n)} \int_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} L^{(n)i_n} L^{(n)j_n} d\theta_3, \tag{30}$$

we arrive at the expression for calculating the stress resultants

$$H_{ij}^{(n)i_n} = \sum_{j_n} \sum_{k,\ell} D_{ijk\ell}^{(n)i_n j_n} \varepsilon_{k\ell}^{(n)j_n}. \tag{31}$$

Note. Indefinite integral (30) can be calculated exactly by using the Gaussian quadrature formulas of an order of accuracy $2I_n - 1$.

Finite-Element Formulation of the Problem

Variational equation (28), with account of Eqs. (30) and (31), is the basis for constructing a finite element of the layered plate. For displacements of the reference surface $\Omega^{(n)i_n}$, we employ the standard bilinear approximation

$$u_i^{(n)i_n} = \sum_r N_r u_{ir}^{(n)i_n}, \tag{32}$$

where $N_r(\xi_1, \xi_2)$ are bilinear shape functions of the finite element; $u_{ir}^{(n)i_n}$ are displacements of the reference surface $\Omega^{(n)i_n}$ at the nodes of the element; ξ_1 and ξ_2 are the normalized coordinates of the finite element; the index *r* shows the node number and varies from 1 to 4.

For strains of the reference surface $\Omega^{(n)i_n}$, we also use the bilinear interpolations [8, 9]

$$\varepsilon_{ij}^{(n)i_n} = \sum_r N_r \varepsilon_{ijr}^{(n)i_n}, \tag{33}$$

where $\varepsilon_{ijr}^{(n)i_n}$ are strains of the reference surface $\Omega^{(n)i_n}$ at element nodes. We should note that approximation (33) makes it possible to use an effective analytical integration within the limits of the finite element [10, 11].

Introducing finite-element approximations (32) and (33) into variational equation (28), (31) and taking into account the new designation for the vector of nodal displacements

$$\mathbf{U} = \left[\mathbf{U}_1^{\mathrm{T}}\ \mathbf{U}_2^{\mathrm{T}}\ \mathbf{U}_3^{\mathrm{T}}\ \mathbf{U}_4^{\mathrm{T}} \right]^{\mathrm{T}}, \tag{34}$$

$$\mathbf{U}_r = \Bigg[\left(\mathbf{u}_r^{[0]}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{(1)2}\right)^{\mathrm{T}} \ldots \left(\mathbf{u}_r^{(1)I_1-1}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{[1]}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{(2)2}\right)^{\mathrm{T}} \ldots$$

$$\left(\mathbf{u}_r^{(N-1)I_{N-1}-1}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{[N-1]}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{(N)2}\right)^{\mathrm{T}} \ldots \left(\mathbf{u}_r^{(N)I_N-1}\right)^{\mathrm{T}} \left(\mathbf{u}_r^{[N]}\right)^{\mathrm{T}} \Bigg]^{\mathrm{T}},$$

$$\mathbf{u}_r^{[m]} = \left[u_{1r}^{[m]}\ u_{2r}^{[m]}\ u_{3r}^{[m]} \right]^{\mathrm{T}} \qquad \left(m = 0, 1, \ldots, N\right),$$

TABLE 1. Calculation Results for the Cylindrical Bending of a Thick Plate at $a/h = 4$

Variant	$U_3(-0.5)$	$U_3(0)$	$U_3(0.5)$	$S_{11}(0.5)$	$S_{12}(0.5)$	$S_{13}(0)$	$S_{23}(0.3)$	$S_{33}(0.5)$
I_n=3	3.192	3.253	3.375	8.396	–4.022	4.366	–1.261	1.032
I_n=4	3.230	3.290	3.413	8.520	–4.080	5.021	–1.263	1.007
I_n=5	3.230	3.290	3.413	8.519	–4.080	5.021	–1.256	1.000

$$\mathbf{u}_r^{(n)m_n} = \left[u_{1r}^{(n)m_n} \; u_{2r}^{(n)m_n} \; u_{3r}^{(n)m_n} \right]^{\mathrm{T}} \qquad \left(m_n = 2,\ldots,I_n - 1 \right),$$

we come to the equilibrium equations for the finite element of the layered plate

$$\mathbf{KU} = \mathbf{F}, \tag{35}$$

where **K** is the rigidity matrix of the finite element, and F is the vector of surface loads.

Numerical Results and Discussion

1. As a first example, we will consider the cylindrical bending of a hinge-supported three-layer composite plate with lay-up [30/–30/30] under a transverse load $p_3^+ = p_0 \sin(\pi\theta_1 / a)$, where a is the plate length. The plate consists of composite layers with the following geometrical and mechanical parameters: $h_1 = h_3 = 0.25$, $h_2 = 0.5$, $E_{\mathrm{L}} = 25E_{\mathrm{T}}$, $G_{\mathrm{LT}} = 0.5E_{\mathrm{T}}$, $G_{\mathrm{TT}} = 0.2E_{\mathrm{T}}$, $E_{\mathrm{T}} = 10^6$, and $\nu_{\mathrm{LT}} = \nu_{\mathrm{TT}} = 0.25$. The subscripts L and T correspond to the reinforcement and transverse directions, respectively. To satisfy the boundary conditions at end faces of the plate, it was assumed that $u_3^{(n)i_n} = 0$, where $n = 1, 2, 3$ and $i_n = 1,2,\ldots,I_n$. For comparison with the analytical solution of the plane problem of elasticity theory [12], we introduce the dimensionless quantities

$$U_3 = 100E_{\mathrm{T}}h^3 u_3(a/2, z)/p_0 a^4,$$
$$S_{11} = 10h^2\sigma_{11}(a/2, z)/p_0 a^2, \qquad S_{12} = 10h^2\sigma_{12}(0, z)/p_0 a^2, \tag{36}$$
$$S_{\alpha 3} = 10h\sigma_{\alpha 3}(0, z)/p_0 a, \qquad S_{33} = \sigma_{33}(a/2, z)/p_0, \qquad z = \theta_3/h.$$

Since the problem is symmetric, we consider half of the plate, which is modeled by 64 finite elements describing the plane stress state of the plate. As shown in Table 1, at an appropriate choice of equidistant reference surfaces, a good agreement with the analytical solution can be achieved even for a thick plate (see also Fig. 3). The distribution of stresses across the thickness of the shell shown in Fig. 3 in the cases of 10 equidistant reference surfaces ($I_1 = I_2 = I_3 = 4$) at $a/h = 10,\ 100$ and 16 reference surfaces ($I_1 = I_2 = I_3 = 6$) at $a/h = 2,\ 4$ also attests to the high potential of the method suggested for solving the problems of statics for layered plates in a 3D statement. As seen, the boundary conditions on surfaces of the plate and the continuity conditions at the interfaces of layers for the transverse tangential stresses are satisfied with an accuracy sufficient in practice.

2. Let us examine a hinge-supported rectangular three-layer composite plate with lay-up [0/90/0] under a sinusoidally distributed load

$$p_3^+ = p_0 \sin(\pi\theta_1 / a)\sin(\pi\theta_2 / b), \tag{37}$$

where *a* and *b* are the length and width of the plate. The geometrical parameters of the plate are $b = 3a$, $h = 1$, and $h_n = h/3$. The mechanical characteristics of the composite are given in the previous example. For comparison with the analytical solution of the 3D problem of elasticity theory [13], we introduce the dimensionless quantities

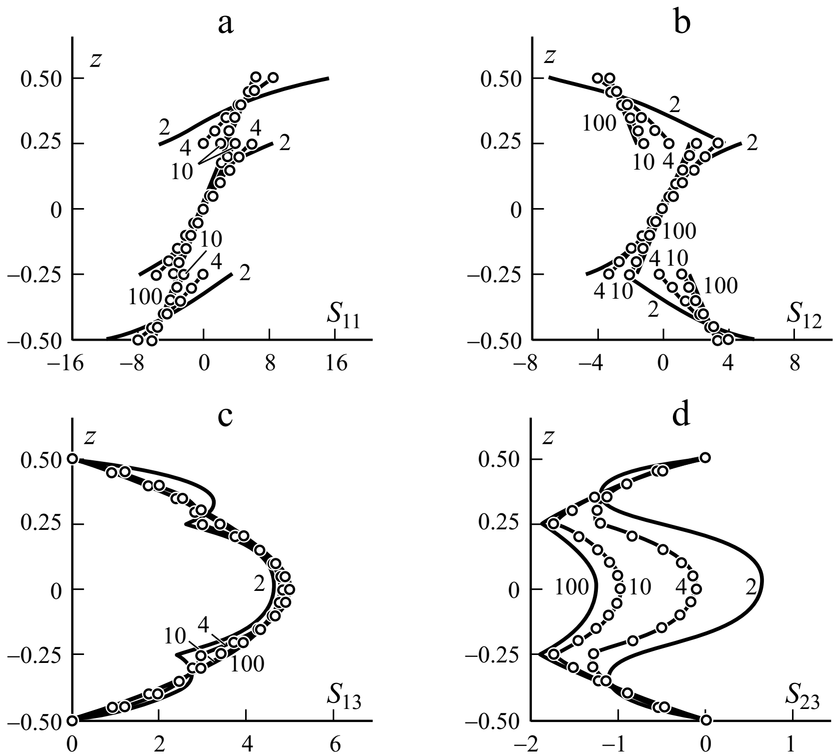

Fig. 3. Distribution of the stresses S_{11} (a), S_{12} (b), S_{13} (c), and S_{23} (d) across the thickness of a plate: exact solution [12] and the given theory of layered plates (○) at different values of a/h (numbers at the curves).

$$
\begin{gathered}
U_3 = 100 E_{\mathrm{T}} h^3 u_3(a/2,\, b/2,\, z)/p_0 a^4, \\
S_{\alpha\alpha} = h^2 \sigma_{\alpha\alpha}(a/2,\, b/2,\, z)/p_0 a^2, \qquad S_{12} = h^2 \sigma_{12}(0,\, 0,\, z)/p_0 a^2, \\
S_{13} = h\sigma_{13}(0,\, b/2,\, z)/p_0 a, \qquad S_{23} = h\sigma_{23}(a/2,\, 0,\, z)/p_0 a, \\
S_{33} = \sigma_{33}(a/2,\, b/2,\, z)/p_0, \qquad z = \theta_3/h.
\end{gathered}
\tag{38}
$$

Owing to symmetry of the problem, a quarter of the plate can be considered. In this case, every reference surface $\Omega^{(n)i_n}$ was modeled with the help of uniform 64 × 64 finite-element meshes. Data in Tables 2 and 3 indicate that a proper choice of equidistant reference surfaces can yield a good agreement with the exact solution of the three-dimensional problem of elasticity theory even for the case of thick plates. The distribution of transverse tangential stresses across the thickness of the shell (Fig. 4) reveals that, in the case of 10 equidistant reference surfaces chosen, the boundary conditions on faces of the plate can be satisfied and the continuity conditions at the interfaces of layers can be fulfilled with an accuracy admissible in engineering calculations.

3. Now, we will examine a hinge-supported square sandwich plate subjected to a load distributed over the face $\Omega^{[3]}$ according to law (37), where $b = a$. The thicknesses of the load-carrying layers and core were chosen as in [13]: $h_1 = h_3 = 0.1$ and $h_2 = 0.8$. The mechanical characteristics of the load-carrying layers are given in the first example; for the core, we selected the following values of mechanical characteristics: $E_1 = E_2 = 4 \cdot 10^4$, $E_3 = 5 \cdot 10^5$, $G_{13} = G_{23} = 6 \cdot 10^4$, $G_{12} = 1.6 \cdot 10^4$, and $\nu_{31} = \nu_{32} = \nu_{12} = 0.25$. For comparison with the analytical solution of the 3D problem of elasticity theory [13], we also introduce the dimensionless quantities (38).

TABLE 2. Calculation Results for a Thick Rectangular Plate at $a/h = 2$

Variant	$U_3(0)$	$S_{11}(0.5)$	$10S_{22}(1/6)$	$10^2S_{12}(0.5)$	$10S_{13}(0)$	$10^2S_{23}(0)$	$S_{33}(0.5)$
I_n=2	7.325	1.578	1.934	–4.917	2.529	5.788	0.879
I_n=3	7.911	1.926	2.217	–5.342	2.441	5.726	1.035
I_n=4	8.168	2.129	2.293	–5.629	2.570	6.680	1.037
[13]	8.17	2.13	2.30	–5.64	2.57	6.68	1.00

TABLE 3. Calculation Results for Thin and Thick Rectangular Plates

Variant	$U_3(0)$	$S_{11}(0.5)$	$10S_{22}(1/6)$	$10^2S_{12}(0.5)$	$10S_{13}(0)$	$10^2S_{23}(0)$	$S_{33}(0.5)$
I_n=2	7.325	1.578	1.934	–4.917	2.529	5.788	0.879
I_n=3	7.911	1.926	2.217	–5.342	2.441	5.726	1.035
I_n=4	8.168	2.129	2.293	–5.629	2.570	6.680	1.037
[13]	8.17	2.13	2.30	–5.64	2.57	6.68	1.00

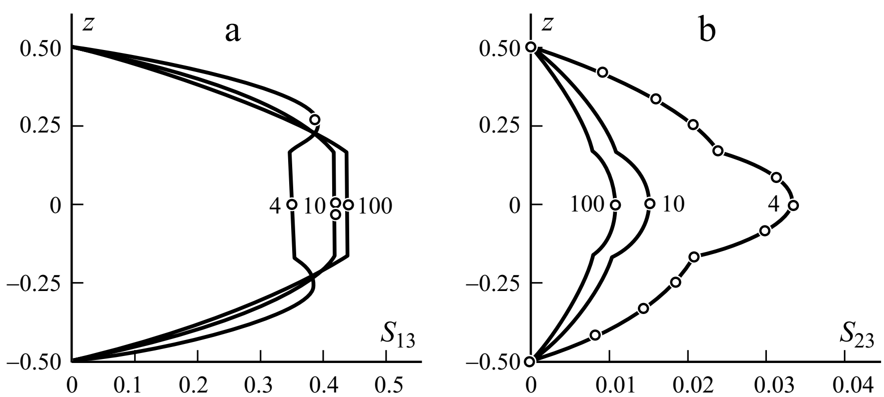

Fig. 4. Distribution of the stresses S_{13} (a) and S_{23} (b) across the thickness of a rectangular plate: exact solution [13] (○) and the given theory of layered plates (——) at various values of a/h (numbers at the curves).

Considering the conditions of symmetry, we examined a quarter of the plate and modeled the reference surfaces $\Omega^{(n)i_n}$ with the help of uniform 64 × 64 finite-element meshes. As seen from Tables 4 and 5, by choosing the equidistant reference surfaces properly, it is possible to come to a satisfactory agreement with the analytical solution both for thick and thin sandwich plates. The distribution of transverse tangential stresses across the plate thickness shown in Fig. 5 also points to a correct satisfaction of boundary conditions on the faces of the plate and of the continuity conditions at the interfaces.

4. Finally, we will consider a hinge-supported sectoral sandwich plate loaded with a uniform pressure $p_3^+ = p_0$ on its face $\Omega^{[3]}$. The plate geometry is shown in Fig. 6a ($b = 5a$, $h_1 = h_3 = 0{,}1$, and $h_2 = 0.8$). The load-carrying layers and the core are made of isotropic materials: $E_f = 10^7$, $\nu_f = 0.3$, $E_c = 10^5$, and $\nu_c = 0.4$.

TABLE 4. Calculation Results for a Square Sandwich Plate at $a/h = 2$

Variant	$S_{11}(0.5)$	$S_{11}(0.4)$	$10S_{22}(0.5)$	$10S_{12}(0.5)$	$10S_{13}(0)$	$10S_{23}(0)$	$S_{33}(0.5)$
I_n=2	3.132	–2.220	4.514	–2.312	1.845	1.379	1.207
I_n=3	3.267	–2.210	4.515	–2.395	1.817	1.354	1.017
I_n=4	3.278	–2.220	4.521	–2.402	1.848	1.399	1.008
[13]	3.278	–2.220	4.517	–2.403	1.85	1.399	1.000

TABLE 5. Calculation Results for Thin and Thick Square Sandwich Plates

a/h	$I_n = 4$				Exact solution [13]			
	$S_{11}(0.5)$	$10S_{12}(0.5)$	$10S_{13}(0)$	$10S_{23}(0)$	$S_{11}(0.5)$	$10S_{12}(0.5)$	$10S_{13}(0)$	$10S_{23}(0)$
4	1.555	–1.436	2.386	1.072	1.556	–1.437	2.39	1.072
10	1.153	–0.706	2.997	0.527	1.153	–0.707	3.00	0.527
100	1.097	–0.436	3.239	0.297	1.098	–0.437	3.24	0.297

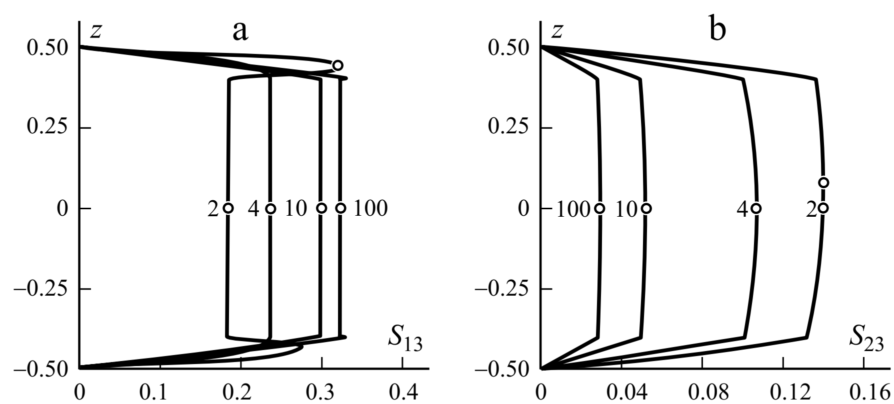

Fig. 5. The same for a square sandwich plate.

Owing to symmetry of the problem, we consider half of the plate and model each reference surface with the use of uniform 64 × 32 finite-element meshes. It is seen that the calculation results for the dimensionless displacement and stresses

$$U_3 = E_c h u_3\left(\left(b-a\right)/2, 30^\circ, z\right)/ p_0 b^2,$$

$$S_{13} = \sigma_{13}\left(b-a, 30^\circ, z\right)/ p_0, \qquad S_{23} = \sigma_{23}\left(\left(b-a\right)/2, 0, z\right)/ p_0,$$

illustrated in Fig. 6 for sections A, B, and C of the plate in the case of 13 equidistant reference surfaces ($I_1 = I_2 = I_3 = 5$), agree well with existing notions about the character of distribution of transverse components of the stress tensor across the thickness of the package.

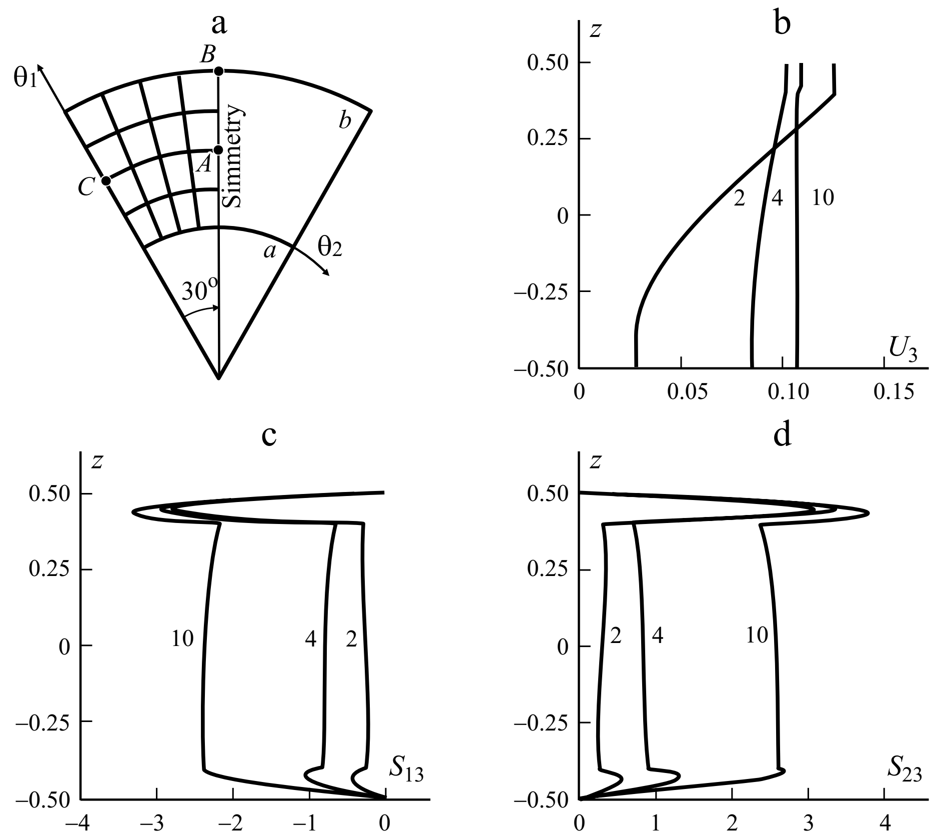

Fig. 6. Geometry of a plate (a) and distributions of the transverse displacement U_3 (b) and of the transverse tangential stresses S_{13} (c) and S_{23} (d) across the thickness of the plate calculated by the given theory of layered plates at different values of b/h (numbers at the curves).

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