ON THE THEORY OF MULTILAYER SHALLOW SHELLS OF FINITE DEFLECTION

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The theory of multilayer shells of finite deflection was formulated in [1,2]. The straight-line hypothesis is assumed to be valid for each layer of the shell. This approach is the most general one, since the transverse shears are described by functions that are arbitrary for each layer, but the order of the equations depends on the number of layers.

Theories based on hypotheses adduced for the entire stack of layers as a whole were considered in [3,4]. A common feature of these studies is that the transverse shear strains are approximated by two functions. The issues in question are treated in greater detail in [5].

In this paper we will allow for transverse shear strain by means of 2r functions that are constant for the entire stack of layers. One particular version of the proposed theory (r = 1) is particularly simple and revealing. For this case we obtain a system of three differential equations in the force function F, displacement function χ, and shear function φ_i^1. Unlike [3,6], the overall order of the system is 12. The equations differ from those for three-layer shells that were obtained in [7] only in terms of constant coefficients.

1. Consider a multilayer shallow shell made up of s transversely isotropic layers. We will disregard transverse squeezing of the layers. The initial surface will be taken to be the internal boundary surface, which we refer to Cartesian coordinates x_1, x_2. The coordinate z will be reckoned in the direction of increase of the outward normal to the initial surface.

Assume that h_k is the thickness of the k-th layer; h is the shell thickness; δ_k is the distance from the initial surface to the end of the k-th layer $(\delta_0=0)$; E_k, ν_k, G_k are the elastic modulus, the Poisson coefficient, and transverse shear modulus of the k-th layer; k_{ij} are the curvatures and torsion of the coordinate lines; u_i and w are the tangential and normal displacements of points of the initial surface; u_i^k are the tangential displacements of points of the k-th layer; α_i^p are functions that characterize the transverse shear; $f_p'(z)$ are known functions (the prime denoting differentiation with respect to z); and δ_{ij} is the Kronecker delta. Here and henceforth $i, j=1, 2;\ k=1, 2, \ldots, s;\ p, q=1, 2, \ldots, r;\ m=0, 1, \ldots, r.$

Following [7], we introduce the adjusted Poisson coefficient ν, the dimensionless layer thicknesses t_k, the dimensionless stiffness characteristics γ_k, the averaged elastic modulus E, and also the necessary notation:

$$\nu=\sum_{k=1}^{s}\frac{E_kh_k\nu_k}{1-\nu_k^2}\left(\sum_{k=1}^{s}\frac{E_kh_k}{1-\nu_k^2}\right)^{-1},\qquad t_k=h_kh^{-1}$$

$$\gamma_k=\frac{E_kh_k}{1-\nu_k^2}\left(\sum_{k=1}^{s}\frac{E_kh_k}{1-\nu_k^2}\right)^{-1},\qquad E=(1-\nu^2)h^{-1}\sum_{k=1}^{s}\frac{E_kh_k}{1-\nu_k^2}$$

$$\omega_k=\delta_kh^{-1},\qquad L_k^p=\sum_{n=1}^{k-1}(f_n^p-f_{n-1}^p)G_n^{-1}-f_{k-1}^pG_k^{-1},\qquad f_n^p=f_p(\delta_n)h^{-1}\ (n=0,1,\ldots,s-1)$$

$$\int_{\delta_{k-1}}^{\delta_k}f_p\,dz=\frac{1}{2}h^2\lambda_k^p,\qquad \int_{\delta_{k-1}}^{\delta_k}f_pz\,dz=\frac{1}{12}h^3\pi_k^p,\qquad \int_{\delta_{k-1}}^{\delta_k}f_p'f_q'\,dz=h\mu_k^{pq},$$

$$\int_{\delta_{k-1}}^{\delta_k} f_p f_q \, dz = \frac{1}{12} h^3 \tau_k^{pq}$$

In what follows we will basically adhere to the notation of [7]. For the k-th layer of the shell we assume that the transverse shear strains are distributed as follows:

$$e_{iz}^k = G_k^{-1} \sum_{p=1}^{r} \alpha_i^p f_p' \tag{1.1}$$

Assume that the layers of the shell operate together without slip, and that the shell is normally loaded by an applied surface load; consequently, the contact conditions and the boundary conditions on the external surfaces can be written as follows:

$$u_i^{n+1}(\delta_n) = u_i^n(\delta_n), \quad \sigma_{iz}^{n+1}(\delta_n) = \sigma_{iz}^n(\delta_n) \quad (n=1,2,\ldots,s-1), \quad \sigma_{iz}^1(0) = 0, \quad \sigma_{iz}^s(h) = 0$$

For these last two conditions to be satisfied, it suffices to set $f_p'(0)=0, \ f_p'(h)=0$. Integrating (1.1) over the thickness of the layers, we can find the tangential displacements of the k-th layer:

$$u_i^k = u_i - \delta_{k-1} w_{,i} - (z - \delta_{k-1}) w_{,i} + \sum_{p=1}^{r} (hL_k^p + G_k^{-1} f_p) \alpha_i^p \tag{1.2}$$

Differentiation with respect to the coordinate x_i is denoted by the subscript after the comma.

The layer strains can be determined from the formulas

$$\varepsilon_{ij}^k = e_{ij} + \delta_{k-1} \varkappa_{ij} + (z - \delta_{k-1}) \varkappa_{ij} + \sum_{p=1}^{r} (hL_k^p + G_k^{-1} f_p) \alpha_{ij}^p$$

$$e_{ij} = {}^1/_2 (u_{i,j} + u_{j,i} + w_{,i} w_{,j}) + k_{ij} w, \qquad \alpha_{ij}^p = {}^1/_2 (\alpha_{i,j}^p + \alpha_{j,i}^p), \qquad \varkappa_{ij} = -w_{,ij}$$

On the basis of Hooke's law, we can write the stresses in the layers as follows:

$$\sigma_{ij}^k = \frac{E_k}{1 - \nu_k^2} [(1 - \nu) \varepsilon_{ij}^k + \nu \delta_{ij} (\varepsilon_{11}^k + \varepsilon_{22}^k)], \qquad \sigma_{iz}^k = G_k e_{iz}^k$$

Using the principle of possible displacements and following [7], we obtain $2r+3$ equilibrium equations for multilayer shallow shells of finite deflection:

$$N_{1i,1} + N_{2i,2} = 0, \qquad H_{1i,i}^q + H_{2i,2}^q = Q_i^q$$

$$M_{11,11} + 2M_{12,12} + M_{22,22} - N_{11}(k_{11} + \varkappa_{11}) - 2N_{12}(k_{12} + \varkappa_{12}) - N_{22}(k_{22} + \varkappa_{22}) + q_0 = 0$$

$$N_{ij} = \sum_{k=1}^{s} N_{ij}^k, \qquad M_{ij} = \sum_{k=1}^{s} (\delta_{k-1} N_{ij}^k + M_{ij}^k), \qquad Q_i^q = \sum_{k=1}^{s} G_k^{-1} Q_i^{qk},$$

$$H_{ij}^q = \sum_{k=1}^{s} (hL_k^q N_{ij}^k + G_k^{-1} H_{ij}^{qk})$$

$$N_{ij}^k = \int_{\delta_{k-1}}^{\delta_k} \sigma_{ij}^k \, dz, \quad M_{ij}^k = \int_{\delta_{k-1}}^{\delta_k} \sigma_{ij}^k (z - \delta_{k-1}) \, dz, \quad H_{ij}^{qk} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{ij}^k f_q \, dz,$$

$$Q_i^{qk} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{iz}^k f_q' \, dz$$

2. Let us now derive the resolvent equations. For this we write the forces and moments as follows:

$$N_{ij} = \frac{Eh}{1 - \nu^2} \Lambda e_{ij}^\circ, \qquad Q_i^q = h \sum_{p=1}^{r} \eta_i^{pq} \alpha_i^p$$

$$M_{ij} = \frac{1}{2} h c_{13} N_{ij} + D_i \left(\eta_3 \Lambda \varkappa_{ij} + \sum_{p=1}^{r} \eta_2^p \Lambda \alpha_{ij}^p \right), \tag{2.1}$$

$$H_{ij}^q = \frac{1}{2} h c_{12}^q N_{ij} + D_i \left(\eta_2^q \Lambda \varkappa_{ij} + \sum_{p=1}^{r} \eta_1^{qp} \Lambda \alpha_{ij}^p \right)$$

$$\Lambda b_{ij} = (1-\nu) b_{ij} + \nu\delta_{ij}(b_{11}+b_{22}), \quad D_1 = Eh^3[12(1-\nu^2)]^{-1}$$

$$e_{ij}^{\circ} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i} + w_{,i}w_{,j}) + k_{ij}w, \quad u_i^{\circ} = u_i - \frac{1}{2}h\left(c_{13}w_{,i} - \sum_{p=1}^{r} c_{12}^p \alpha_i^p\right)$$

$$c_{12}^p = \sum_{k=1}^{s} (2L_k^p + \lambda_k^p t_k^{-1} G_k^{-1})\gamma_k, \quad c_{13} = \sum_{k=1}^{s} (\omega_k + \omega_{k-1})\gamma_k$$

$$\eta_1^{pq} = \sum_{k=1}^{s} [\gamma_k \tau_k^{pq} t_k^{-1} G_k^{-2} + 6(\lambda_k^q L_k^p + \lambda_k^p L_k^q)\gamma_k t_k^{-1} G_k^{-1} + 12\gamma_k L_k^p L_k^q] - 3c_{12}^p c_{12}^q \tag{2.1}$$

$$\eta_i^{pq} = \sum_{k=1}^{s} \mu_k^{pq} G_k^{-1}, \quad \eta_2^p = \sum_{k=1}^{s} [\gamma_k \pi_k^p t_k^{-1} G_k^{-1} + 6(\omega_k + \omega_{k-1})\gamma_k L_k^p] - 3c_{12}^p c_{13}$$

$$\eta_3 = 4\sum_{k=1}^{s} (t_k^2 + 3\omega_k\omega_{k-1})\gamma_k - 3c_{13}^2$$

Introducing the functions F, φ^p, a^p in a manner similar to [2], we obtain the system of resolvent equations, of which r equations describe the shear boundary effect:

$$\nabla^2\nabla^2 F = Eh(k_{11}w_{,22} - 2k_{12}w_{,12} + k_{22}w_{,11} + w_{,12}^2 - w_{,11}w_{,22}) \tag{2.2}$$

$$D_1 \sum_{p=1}^{r} \nabla^2(\eta_1^{pq} a^p - r^{-1}\eta_2^q w) = h\sum_{p=1}^{r} \eta_i^{pq} a^p \tag{2.3}$$

$$\frac{1-\nu}{2} D_1 \sum_{p=1}^{r} \eta_1^{pq} \nabla^2 \varphi^p = h \sum_{p=1}^{r} \eta_i^{pq} \varphi^p \tag{2.4}$$

$$D_1 \sum_{p=1}^{r} \nabla^2\nabla^2(\eta_2^p a^p - r^{-1}\eta_3 w) - F_{,22}(k_{11} - w_{,11}) + 2F_{,12}(k_{12} - w_{,12}) - F_{,11}(k_{22} - w_{,22}) = -q_0 \tag{2.5}$$

$$N_{ij} = \delta_{ij}\nabla^2 F - F_{,ij}, \quad \alpha_1^p = a_{,1}^p + \varphi_{,2}^p, \quad \alpha_2^p = a_{,2}^p - \varphi_{,1}^p$$

where ∇^2 denotes the Laplace operator.

It is possible to further simplify the equations in (2.2)-(2.5) if we introduce the displacement function χ in accordance with the following formulas:

$$w = \sum_{m=0}^{r} b_m^{\circ} \underbrace{\nabla^2 \ldots \nabla^2}_{m} \chi, \quad a^p = \sum_{m=0}^{r} b_m^p \underbrace{\nabla^2 \ldots \nabla^2}_{m} \chi \tag{2.6}$$

Substituting (2.6) into (2.3) and assuming that the resultant equations are satisfied identically, we obtain a system of linear algebraic equations for determining the coefficients b_m°, b_m^p.

In setting up the theory of multilayer shallow shells of small deflection, we can satisfy Eq. (2.2) identically if we set

$$\chi = \nabla^2\nabla^2\Phi, \quad F = Eh\Omega \sum_{m=0}^{r} b_m^{\circ} \underbrace{\nabla^2 \ldots \nabla^2}_{m} \Phi \tag{2.7}$$

Substituting w and a^p from (2.6) into (2.5), and then χ and F on the basis of (2.7), we obtain an equation of order 2r + 8 in the function Φ:

$$D \sum_{m=0}^{r} \sum_{\mu=1}^{r} \theta_m^p \underbrace{\nabla^2 \ldots \nabla^2}_{m+4} \Phi + Eh\Omega\Omega \sum_{m=0}^{r} b_m^{\circ} \underbrace{\nabla \ldots \nabla^2}_{m} \Phi = q_0 \tag{2.8}$$

$$\Omega = k_{11}\frac{\partial^2}{\partial x_2^2} - 2k_{12}\frac{\partial^2}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2}{\partial x_1^2}, \quad D = D_1\eta_3, \quad \theta_m^p = r^{-1}b_m^{\circ} - \eta_2^p \eta_3^{-1} b_m^p$$

where D is the shell stiffness, corresponding to a surface at a distance $\frac{1}{2}hc_{13}$ from the initial surface.

System (2.4), (2.8) is the resolvent, since all the functions characterizing the stress-strain state of the shell are expressed in terms of φ^p, Φ.

We should note that the equations in (2.4) are unrelated to (2.8) and have a solution of boundary-effect type. This fact makes it possible to approximately set $\varphi^{r}=0$ and thus to lower the order of the resolvent equations by 2r.

The structure of the boundary conditions does not differ markedly from [7], and therefore they will not be given here. We will note only that in the general case we have 4r + 8 boundary conditions, this corresponding to the order of the resolvent equations.

3. Equations (2.2)-(2.5) become particularly transparent if we assume that the transverse shear strains are taken into account by the two functions α_1^1, α_2^1. Then we obtain the following formulas from (2.6):

$$w=\left(1-\frac{h^2}{\beta}\nabla^2\right)\chi,\quad a^1=-\frac{\eta_2^{1}}{\eta_1^{11}}\frac{h^2}{\beta}\nabla^2\chi,\quad \beta=\frac{12(1-\nu^2)}{E}\frac{\eta_4^{11}}{\eta_1^{11}}$$

The strain compatibility equation remains unaltered; the others reduce to the form

$$D\left(1-\frac{\theta h^2}{\beta}\nabla^2\right)\nabla^2\nabla^2\chi+F_{,22}(k_{11}-w_{,11})-2F_{,12}(k_{12}-w_{,12})+F_{,11}(k_{22}-w_{,22})=q_0$$
$$\frac{1-\nu}{2}\frac{h^2}{\beta}\nabla^2\varphi^1=\varphi^1,\quad \theta=1-(\eta_2^{1})^2(\eta_1^{11}\eta_3)^{-1} \tag{3.1}$$

Equations (2.2) and (3.1) and the boundary conditions coincide with the resolvent equations and boundary conditions for three-layer shells, respectively [7]. Thus, we can assert the following: the problems of stability and vibration of three-layer shells, which were solved in [7], can be successfully applied in calculating multilayer shells.

In concluding, we should note that Eqs. (2.2) and (3.1) admit passages to the limit. If we formally set the parameter $\theta=0$, we obtain the resolvent equations from [3]. Taking $G_k=\infty$, we can obtain equations for multilayer shells for which the hypothesis of the nondeforming normal, taken for the entire stack of layers, is valid.

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