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Solution of Three-Dimensional Problems for Thick Elastic Shells by the Method of Reference Surfaces

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Abstract—A new method for solving the elasticity problem for thick and thin shells is proposed. The method is based on the concept of reference surfaces inside the shell. According to this method, N reference surfaces are introduced in the body of the shell so that they are parallel to the midsurface and located at the Chebyshev polynomial nodes, which permits taking the displacement vectors $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N$ of these surfaces for the desired functions. This choice of the desired functions allows one to represent the resolving equations of the proposed theory of higher-order shells in a sufficiently concise form and obtain deformation relations which permit describing the shell displacements as motions of a rigid body.

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1. INTRODUCTION

In the traditional construction of shell theory, displacements are expanded in power series in the transverse coordinate θ_3 measured along the outer normal to the medium surface. For an approximate representation of the displacement field, one can use finite segments of the power series, because the main goal in the theory of elastic shells is to obtain approximate solutions of three-dimensional elasticity problems. The idea of this approach goes back to Cauchy [1]. But the seeming advantage of this theory is lost in static problems for thick elastic shells, where one has to retain very many terms of the expansion to obtain reasonable results.

In a more efficient approach, reference surfaces $\Omega^1, \Omega^2, \ldots, \Omega^N$ parallel to the medium surface are introduced in the shell body so as to use the displacement vectors of these surfaces as the unknown functions [2, 3]. This choice of the unknown functions with the subsequent use of Lagrange polynomials of degree $N-1$ in the spatial approximations to the displacements allows one to represent the resolving equations of the proposed theory of higher-order shells in a sufficiently concise form and construct deformation relations whose exactly describe the shell displacements as motions of a rigid body in a system of curvilinear surface coordinates. But no proof of this fundamental statement has been given. We also note that the idea of the method of reference surfaces goes back to [4–9], where various versions of geometrically linear and nonlinear shell theories were constructed by using the shell face surfaces Ω^- and Ω^+ as the reference surfaces.

The theory of higher-order shells [2, 3] is based on the use of equidistant reference surfaces, and the face surfaces are also chosen as references surfaces. This hinders the application of this theory to thick shells. The point is that, because of the Runge phenomenon, the proposed spatial polynomial interpolation of the displacement vector with the use of Lagrange polynomials of higher degrees can lead to significant oscillations of the polynomial approximations in the boundary effect region. This phenomenon was discovered in [10] when studying the polynomial interpolation error in the approximation of several functions on a uniform grid. The interpolation error can tend to infinity as the polynomial degree increases. In numerical analysis, this effect is usually suppressed by taking the roots of the Chebyshev polynomial [11] for the interpolation nodes, which allows significantly improving the behavior

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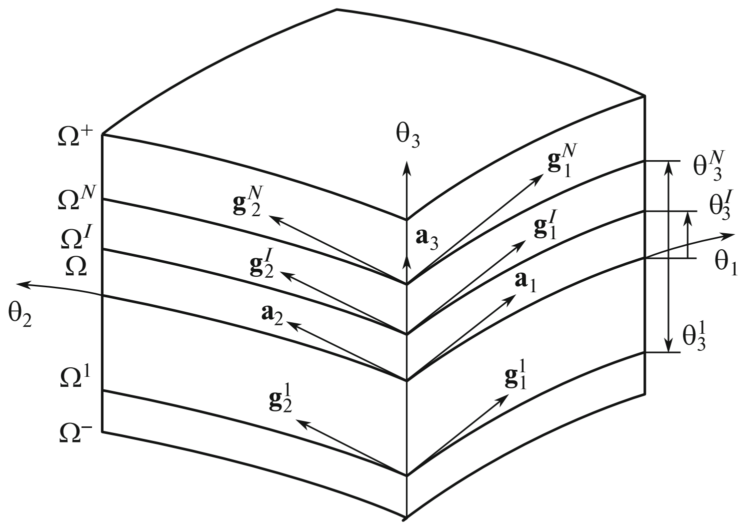

Fig. 1.

of higher-degree polynomial approximations, for which the interpolation error tends to zero as $N \to \infty$. This permits determining the solution of three-dimensional static problems for thick shells with any prescribed accuracy for sufficiently many reference surfaces.

2. SHELL KINEMATICS

We consider a shell of constant thickness h. We refer the midsurface Ω to curvilinear orthogonal coordinates θ_1, θ_2 counted along the principal curvature lines, while the coordinate θ_3 is counted in the transverse direction. The basis vectors of the shell medium surface have the form

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = A_\alpha \mathbf{e}_\alpha, \quad \mathbf{a}_3 = \mathbf{e}_3, \tag{2.1}$$

where $\mathbf{r} = \mathbf{r}(\theta_1, \theta_2)$ is the position vector of the medium surface, the $A_\alpha(\theta_1, \theta_2)$ are the coefficients of the first quadratic form, the $\mathbf{e}_\alpha(\theta_1, \theta_2)$ are the unit tangent vectors to the coordinate lines θ_α, and $\mathbf{e}_3(\theta_1, \theta_2)$ is the unit outward normal vector to the medium surface. Note that the convention concerning summation over repeated indices is not used in the present paper.

Let $\mathbf{R} = \mathbf{r} + \theta_3 \mathbf{e}_3$ be the shell position vector; then the basis vectors in the shell body can be represented as

$$\mathbf{g}_\alpha = \mathbf{R}_{,\alpha} = A_\alpha c_\alpha \mathbf{e}_\alpha, \quad \mathbf{g}_3 = \mathbf{R}_{,3} = \mathbf{e}_3, \tag{2.2}$$

where the $c_\alpha = 1 + k_\alpha \theta_3$ are the components of the geometric shear tensor and the k_α are the principal curvatures.

By $\mathbf{R}^I = \mathbf{r} + \theta_3^I \mathbf{e}_3$ we denote the position vectors of the reference surfaces Ω^I which lie inside the interval $(-h/2, h/2)$ at the nodes of the Chebyshev polynomial of degree N, where θ_3^I are the transverse coordinates of the surfaces Ω^I defined according to [11] by the formula

$$\theta_3^I = -\frac{h}{2} \cos\left(\pi \frac{2I-1}{2N}\right). \tag{2.3}$$

Then the basis vectors of the reference surfaces shown in Fig. 1 have the form

$$\mathbf{g}_\alpha^I = \mathbf{R}_{,\alpha}^I = A_\alpha c_\alpha^I \mathbf{e}_\alpha, \quad \mathbf{g}_3^I = \mathbf{e}_3, \tag{2.4}$$

where the $c_\alpha^I = 1 + k_\alpha \theta_3^I$ are the components of the geometric shear tensor on the surfaces Ω^I.

The basis vectors in the shell body in a strained state are determined by the formulas

$$\bar{\mathbf{g}}_i = \bar{\mathbf{R}}_{,i} = \mathbf{g}_i + \mathbf{u}_{,i}, \tag{2.5}$$

where $\bar{\mathbf{R}} = \mathbf{R} + \mathbf{u}$ is the position vector of the deformed shell and $\mathbf{u}$ is the displacement vector.

The basis vectors of the reference surfaces Ω^I in the strained state can be represented as

$$\bar{\mathbf{g}}_\alpha^I = \bar{\mathbf{R}}_{,\alpha}^I = \mathbf{g}_\alpha^I + \mathbf{u}_{,\alpha}^I, \quad \bar{\mathbf{g}}_3^I = \bar{\mathbf{g}}_3(\theta_3^I) = \mathbf{e}_3 + \boldsymbol{\beta}^I, \tag{2.6}$$

$$\mathbf{u}^I = \mathbf{u}(\theta_3^I), \quad \boldsymbol{\beta}^I = \mathbf{u}_{,3}(\theta_3^I), \tag{2.7}$$

where the $\bar{\mathbf{R}}^I = \mathbf{R}^I + \mathbf{u}^I$ are the position vectors of the surfaces Ω^I in the strained state, the $\mathbf{u}^I(\theta_1, \theta_2)$ are the displacement vectors of the surfaces Ω^I, and the $\boldsymbol{\beta}^I(\theta_1, \theta_2)$ are the values of the derivative of the three-dimensional displacement vector with respect to the coordinate θ_3 on the surfaces Ω^I. From now on, the indices I, J, K indicate that a certain variable pertains to a reference surface and take the values $1, 2, \ldots, N$, the Greek indices are $\alpha, \beta = 1, 2$, and the Latin indices are $i, j, k, m = 1, 2, 3$.

3. DEFORMATION RELATIONS

In the system of curvilinear orthogonal coordinates θ_i, the strain tensor can be written as [12, 13]

$$2\varepsilon_{ij} = \frac{1}{A_i A_j c_i c_j}(\bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j - \mathbf{g}_i \cdot \mathbf{g}_j), \tag{3.1}$$

where $A_3 = 1$ and $c_1 = 1$. The values of the strain tensor components on the reference surfaces Ω^I have the form

$$2\varepsilon_{ij}^I = 2\varepsilon_{ij}(\theta_3^I) = \frac{1}{A_i A_j c_i^I c_j^I}(\bar{\mathbf{g}}_i^I \cdot \bar{\mathbf{g}}_j^I - \mathbf{g}_i^I \cdot \mathbf{g}_j^I). \tag{3.2}$$

By introducing the basis vectors (2.4) and (2.6) into the deformation relations (3.2) of 3D elasticity and by omitting the nonlinear terms, we obtain

$$\begin{gathered} 2\varepsilon_{\alpha\beta}^I = \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_\beta + \frac{1}{A_\beta c_\beta^I}\mathbf{u}_{,\beta}^I \cdot \mathbf{e}_\alpha, \\ 2\varepsilon_{\alpha 3}^I = \boldsymbol{\beta}^I \cdot \mathbf{e}_\alpha + \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_3, \quad \varepsilon_{33}^I = \boldsymbol{\beta}^I \cdot \mathbf{e}_3. \end{gathered} \tag{3.3}$$

Further, we represent the vectors $\mathbf{u}^I$ and $\boldsymbol{\beta}^I$ in the orthonormal basis $\mathbf{e}_i$ by the formulas

$$\mathbf{u}^I = \sum_i u_i^I \mathbf{e}_i, \tag{3.4}$$

$$\boldsymbol{\beta}^I = \sum_i \beta_i^I \mathbf{e}_i. \tag{3.5}$$

From expansion (3.4) with the formulas of differentiation of the basis vectors $\mathbf{e}_i$ with respect to the curvilinear orthogonal coordinates [14]

$$\begin{gathered} \frac{1}{A_\alpha}\mathbf{e}_{\alpha,\alpha} = -B_\alpha \mathbf{e}_\beta - k_\alpha \mathbf{e}_3, \quad \frac{1}{A_\alpha}\mathbf{e}_{\beta,\alpha} = B_\alpha \mathbf{e}_\alpha, \\ \frac{1}{A_\alpha}\mathbf{e}_{3,\alpha} = k_\alpha \mathbf{e}_\alpha, \quad B_\alpha = \frac{1}{A_\alpha A_\beta}A_{\alpha,\beta} \quad (\beta \neq \alpha) \end{gathered} \tag{3.6}$$

taken into account, we obtain

$$\frac{1}{A_\alpha}\mathbf{u}_{,\alpha}^I = \sum_i \lambda_{i\alpha}^I \mathbf{e}_i, \tag{3.7}$$

$$\lambda_{\alpha\alpha}^I = \frac{1}{A_\alpha}u_{\alpha,\alpha}^I + B_\alpha u_\beta^I + k_\alpha u_3^I, \quad \lambda_{\beta\alpha}^I = u_{\beta,\alpha}^I - B_\alpha u_\alpha^I \quad (\beta \neq \alpha), \quad \lambda_{3\alpha}^I = \frac{1}{A_\alpha}u_{3,\alpha}^I - k_\alpha u_\alpha^I. \tag{3.8}$$

By substituting the expansions (3.5) and (3.7) into formulas (3.3), we obtain the scalar form of the linearized deformation relations

$$2\varepsilon_{\alpha\beta}^I = \frac{1}{c_\beta^I}\lambda_{\alpha\beta}^I + \frac{1}{c_\alpha^I}\lambda_{\beta\alpha}^I, \quad 2\varepsilon_{\alpha 3}^I = \beta_\alpha^I + \frac{1}{c_\alpha^I}\lambda_{3\alpha}^I, \quad \varepsilon_{33}^I = \vartheta_3^I. \tag{3.9}$$

4. SPATIAL APPROXIMATIONS TO DISPLACEMENTS AND STRAINS

Note that no assumptions about the character of the displacement and strain distribution over the shell thickness have been made so far. We assume that the displacements are distributed across the shell thickness according to the law [2]

$$u_i = \sum_I L^I u_i^I, \tag{4.1}$$

where the $L^I(\theta_3)$ are the Lagrange polynomials of degree $N-1$ defined by the formula

$$L^I = \prod_{J \neq I} \frac{\theta_3 - \theta_3^J}{\theta_3^I - \theta_3^J}. \tag{4.2}$$

In this case, $L^I(\theta_3^J) = 1$ if $J = I$ and $L^I(\theta_3^J) = 0$ if $J \neq I$.

It follows from relations (2.7), (3.5), (4.1), and (4.2) that

$$\beta_i^I = \sum_J M^J(\theta_3^I) u_i^J, \tag{4.3}$$

where the $M^I = L^I_{,3}$ are polynomials of degree $N-2$; their values on the reference surfaces Ω^I are determined by the formulas

$$\begin{aligned} M^J(\theta_3^I) &= \frac{1}{\theta_3^J - \theta_3^I} \prod_{K \neq I,J} \frac{\theta_3^I - \theta_3^K}{\theta_3^J - \theta_3^K} \quad (J \neq I), \\ M^I(\theta_3^I) &= -\sum_{J \neq I} M^J(\theta_3^I). \end{aligned} \tag{4.4}$$

Thus, the determining functions β_i^I of the proposed shell theory are represented as a linear combination of displacements u_i^J of the reference surfaces.

The next step is to choose the strain distribution law across the shell thickness. It is obvious that the strain distribution in the transverse direction must be consistent with the displacement distribution (4.1); i.e., we have

$$\varepsilon_{ij} = \sum_I L^I \varepsilon_{ij}^I. \tag{4.5}$$

Theorem. *Deformation relations* (3.3), (4.5) *exactly represent the shell displacement as motions of a rigid body in the system of curvilinear spatial coordinates.*

Proof. The displacement of the reference surfaces Ω^I as motions of a rigid body can be represented as [15, 16]

$$(\mathbf{u}^I)^{\text{Rigid}} = \mathbf{\Delta} + \mathbf{\Phi} \times \mathbf{R}^I, \tag{4.6}$$

$$\mathbf{\Delta} = \sum_i \Delta_i \mathbf{e}_i, \quad \mathbf{\Phi} = \sum_i \Phi_i \mathbf{e}_i, \tag{4.7}$$

where $\mathbf{\Delta}$ is the shell translational motion vector and $\mathbf{\Phi}$ is the rotation vector. According to [14], we have

$$\mathbf{\Delta}_{,\alpha} = \mathbf{0}, \quad \mathbf{\Phi}_{,\alpha} = \mathbf{0}. \tag{4.8}$$

It follows from (4.6) with (2.1), (3.6), and (4.8) taken into account that

$$(\mathbf{u}^I_{,\alpha})^{\text{Rigid}} = A_\alpha c_\alpha^I \mathbf{\Phi} \times \mathbf{e}_\alpha. \tag{4.9}$$

By taking into account the identities

$$\sum_J M^J(\theta_3) = 0, \quad \sum_J \theta_3^J M^J(\theta_3) = 1 \tag{4.10}$$

which, in turn, follow from the obvious identities

$$\sum_J L^J(\theta_3) = 1, \quad \sum_J \theta_3^J L^J(\theta_3) = \theta_3 \tag{4.11}$$

and by considering (2.7), (4.1), and (4.6), we obtain

$$(\boldsymbol{\beta}^I)^{\text{Rigid}} = \sum_J M^J(\theta_3^I)(\mathbf{u}^J)^{\text{Rigid}} = \boldsymbol{\Phi} \times \mathbf{e}_3. \tag{4.12}$$

By introducing (4.9) and (4.12) into the deformation relations (3.3), we arrive at

$$2(\varepsilon_{ij}^I)^{\text{Rigid}} = (\boldsymbol{\Phi} \times \mathbf{e}_i)\mathbf{e}_j + (\boldsymbol{\Phi} \times \mathbf{e}_j)\mathbf{e}_i = 0, \tag{4.13}$$

as required.

5. TOTAL POTENTIAL ENERGY

We substitute the deformations (4.5) into the expression for the total potential energy of the elastic body [17], introduce the resultant stresses [2]

$$H_{ij}^I = \int_{-h/2}^{h/2} \sigma_{ij} L^I c_1 c_2 \, d\theta_3, \tag{5.1}$$

and obtain

$$\Pi = \iint_\Omega \left[\frac{1}{2} \sum_I \sum_{i,j} H_{ij}^I \varepsilon_{ij}^I - \sum_i (c_1^+ c_2^+ p_i^+ u_i^+ - c_1^- c_2^- p_i^- u_i^-) \right] A_1 A_2 \, d\theta_1 \, d\theta_2 - W_\Sigma, \tag{5.2}$$

where p_i^- and p_i^+ are surface loads acting on the shell outer and inner surfaces Ω^- and Ω^+, $u_i^- = u_i(-h/2)$ and $u_i^+ = u_i(h/2)$ are the displacements of the surfaces Ω^- and Ω^+, $c_\alpha^- = 1 - k_\alpha h/2$ and $c_\alpha^+ = 1 + k_\alpha h/2$ are components of the geometric shear tensor on the surfaces Ω^- and Ω^+, and W_Σ is the work of the external loads acing on the shell lateral surface Σ.

We restrict our consideration to the case of linearly elastic materials obeying the generalized Hooke law

$$\sigma_{ij} = \sum_{k,m} C_{ijkm} \varepsilon_{km}, \tag{5.3}$$

where C_{ijkm} is the tensor of elastic moduli.

We substitute the stresses (5.3) into (5.1), take into account the strain distribution in the transverse direction, and obtain the formula for calculating the resultant stresses,

$$H_{ij}^I = \sum_J \sum_{k,m} D_{ijkm}^{IJ} \varepsilon_{km}^J, \tag{5.4}$$

$$D_{ijkm}^{IJ} = C_{ijkm} \int_{-h/2}^{h/2} L^I L^J c_1 c_2 \, d\theta_3. \tag{5.5}$$

6. NUMERICAL RESULTS

As an example, consider the bending of a hinged short cylindrical shell of dimension $L/R = 4$ loaded on the inner surface Ω^- by the sinusoidally distributed load

$$p_3^- = -p_0 \sin \frac{\pi \theta_1}{L} \cos(4\theta_2), \tag{6.1}$$

Table 1

N	$U_3(0)$	$S_{11}(0.5)$	$S_{22}(0.5)$	$S_{12}(-0.5)$	$S_{13}(0)$	$S_{23}(0)$	$S_{33}(-0.5)$
6	7.272	0.914	4.290	−1.597	1.514	−2.128	−1.145
7	7.483	1.167	5.042	−1.741	1.496	−1.987	−1.069
9	7.501	1.287	5.141	−1.759	1.503	−2.073	−1.020
11	7.503	1.324	5.159	−1.761	1.504	−2.052	−1.005
13	7.503	1.332	5.163	−1.761	1.504	−2.056	−1.001
[18]	7.503	1.332	5.163	−1.761	1.504	−2.056	−1.000

Table 2

N	$U_3(0)$	$S_{11}(0.5)$	$S_{22}(0.5)$	$S_{12}(-0.5)$	$S_{13}(0)$	$S_{23}(0)$	$S_{33}(-0.5)$
6	7.249	0.937	4.413	−1.583	1.508	−2.125	−1.139
7	7.466	1.202	5.065	−1.730	1.496	−1.983	−1.067
9	7.498	1.354	5.166	−1.756	1.498	−2.065	−1.025
11	7.509	1.439	5.234	−1.764	1.511	−2.068	−1.018
13	7.531	1.520	5.324	−1.773	1.501	−2.041	−1.026
[18]	7.503	1.332	5.163	−1.761	1.504	−2.056	−1.000

where L is the shell length, R is the medium surface radius, and θ_1, θ_2 are the meridional and circular coordinates of the shell. The shell is manufactured of a composite material with the following characteristics [18]: $E_{\mathrm{L}} = 25E_{\mathrm{T}}$, $G_{\mathrm{LT}} = 0.5E_{\mathrm{T}}$, $G_{\mathrm{TT}} = 0.2E_{\mathrm{T}}$, $E_{\mathrm{T}} = 10^6$, and $\nu_{\mathrm{LT}} = \nu_{\mathrm{TT}} = 0.25$. The subscripts 'L' and 'T' correspond to the reinforcement direction and the transverse direction. The reinforcing filaments lie in the circular direction.

To satisfy the boundary conditions, we assume that

$$u_1^I = u_{10}^I \cos\frac{\pi\theta_1}{L}\cos(4\theta_2), \quad u_2^I = u_{20}^I \sin\frac{\pi\theta_1}{L}\sin(4\theta_2), \quad u_3^I = u_{30}^I \sin\frac{\pi\theta_1}{L}\cos(4\theta_2). \tag{6.2}$$

We substitute the displacements (6.2) into the formula for the total potential energy (5.2), take into account relations (3.8), (3.9), (4.3), (5.4), and $W_\Sigma = 0$, and obtain

$$\Pi = \Pi(u_{i0}^I). \tag{6.3}$$

Further, we apply the principle of minimum of the total potential energy and obtain a system of algebraic equations of order N,

$$\frac{\partial \Pi}{\partial u_{i0}^I} = 0. \tag{6.4}$$

The above-described algorithm was implemented in the programming environment MATLAB using the package ToolBox Symbolic Math, which allows one to perform symbolic calculations. As a result, we obtain an analytic solution of the problem on the basis of the considered theory of higher-order shells. To compare the results with the analytic solution of the three-dimensional elasticity problem [18], we introduce the three-dimensional variables

$$\begin{gathered} S_{11} = \frac{100h^2\sigma_{11}(L/2,0,z)}{p_0R^2}, \quad S_{22} = \frac{10h^2\sigma_{22}(L/2,0,z)}{p_0R^2}, \quad S_{12} = \frac{100h^2\sigma_{12}(0,\pi/8,z)}{p_0R^2}, \\ S_{13} = \frac{100h\sigma_{13}(0,0,z)}{p_0R}, \quad S_{23} = \frac{10h\sigma_{23}(L/2,\pi/8,z)}{p_0R}, \quad S_{33} = \frac{\sigma_{33}(L/2,0,z)}{p_0}, \\ U_3 = \frac{10E_{\mathrm{L}}h^3u_3(L/2,0,z)}{p_0R^4}, \quad z = \frac{\theta_3}{h}. \end{gathered} \tag{6.5}$$

The data in Table 1, which were obtained by using the reference surfaces at the nodes of the Chebyshev polynomial, and the data in Table 2, which were obtained by using the equidistant reference

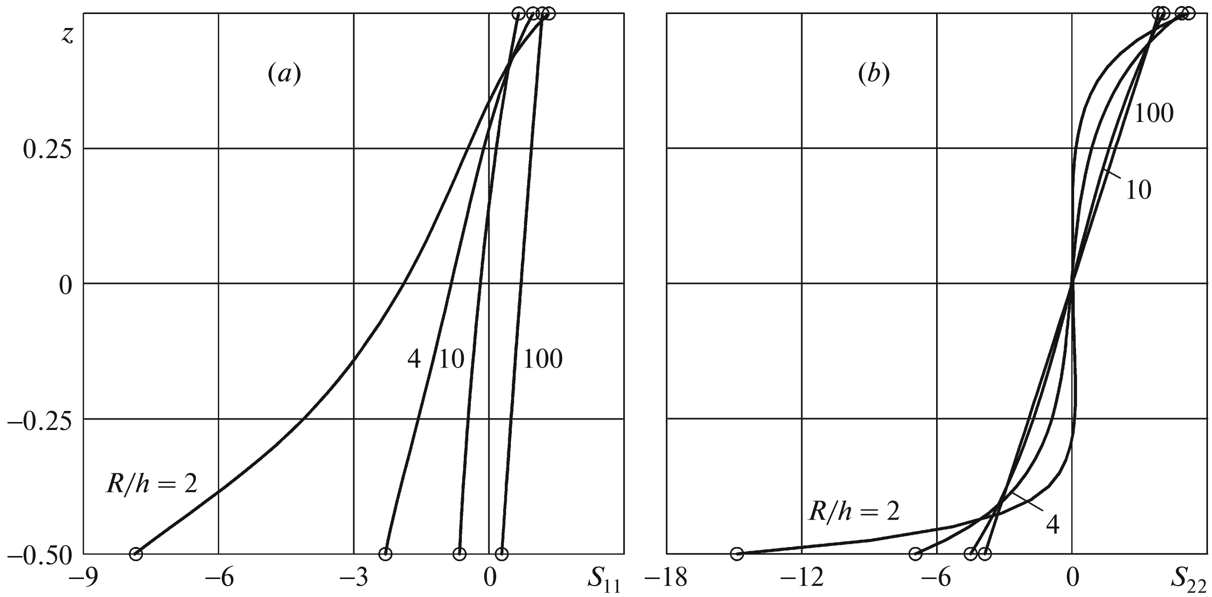

Fig. 2.

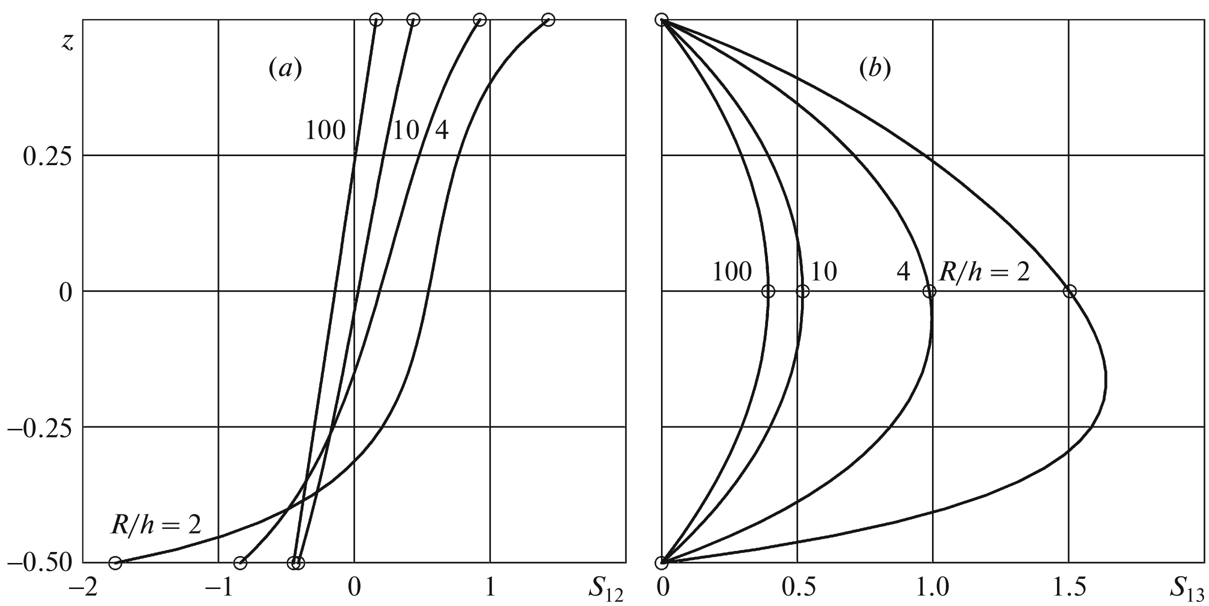

Fig. 3.

surfaces [3], show that, for a sufficiently large N, one can obtain a good agreement with the analytic solution of the elasticity problem [18] even for a thick shell with $R/h = 2$. Note that the approach in [3] leads to rather good results for $N = 9$; the choice of a greater number of reference surfaces does not improve the results of calculations, because there is no uniform convergence to the exact solution of the problem. The distribution of dimensionless stresses S_{ij} over the shell thickness, which is shown in Figs. 2–4 in the case of 11 reference surfaces for shells with geometric parameters $R/h = 2, 4, 10, 100$, also testifies that the proposed method for solving static problems for thick and thin shells in the 3D statement is highly promising. The solid curves were obtained by using this approach, and the small circles present the results obtained in [18]. We can see that the boundary conditions on the shell face surfaces for the transverse components of the stress tensor are satisfied with a sufficiently high accuracy.

Figures 5–7 additionally show the logarithmic error $\delta_i = \lg|S_{i3}^{3D} - S_{i3}|$ in the boundary conditions for these stresses on the inner surface (curves with small circles) and the outer surface (curves with small

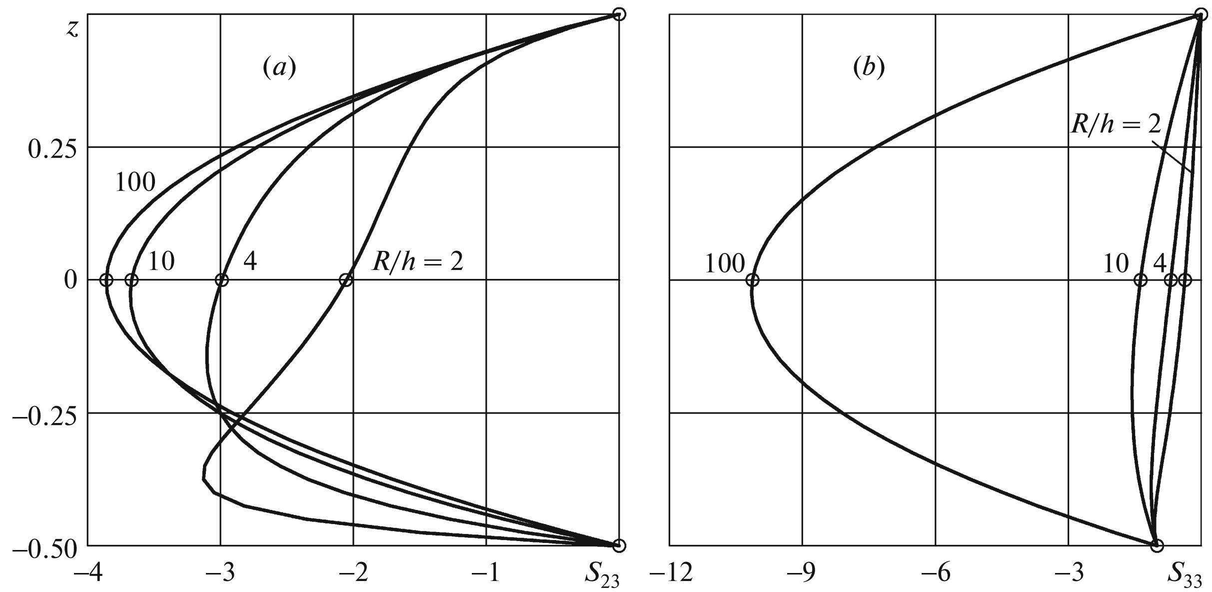

Fig. 4.

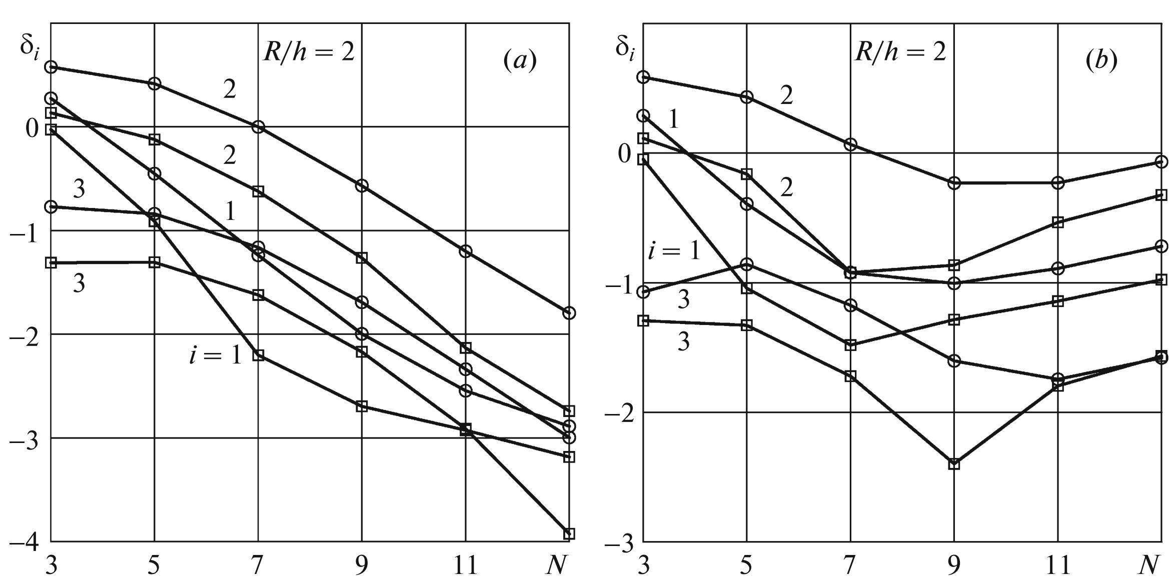

Fig. 5.

squares) for various values of the parameter N. Figures 5 *a*, 6 *a*, 7 *a* illustrate the results of solution of the problem by the proposed method, Figs. 5 *b*, 6 *b*, 7 *b* show the results obtained in [3]. As was already noted, the calculations based on the use of equidistant surfaces [3] do not ensure the monotone convergence of the solution and give an inadequate description of the shell stress state in the boundary effect region for interpolation polynomials of high degree.

7. CONCLUSION

A new method for solving 3D elasticity problems for thick and thin shells is proposed. According to this method, several reference surfaces are introduced inside the shell at the nodes of the Chebyshev polynomials so as to take the displacement vectors of these surfaces for the unknown functions. It is shown that the solutions of three-dimensional static problems for elastic shells can be found with any prescribed accuracy for sufficiently many reference surfaces.

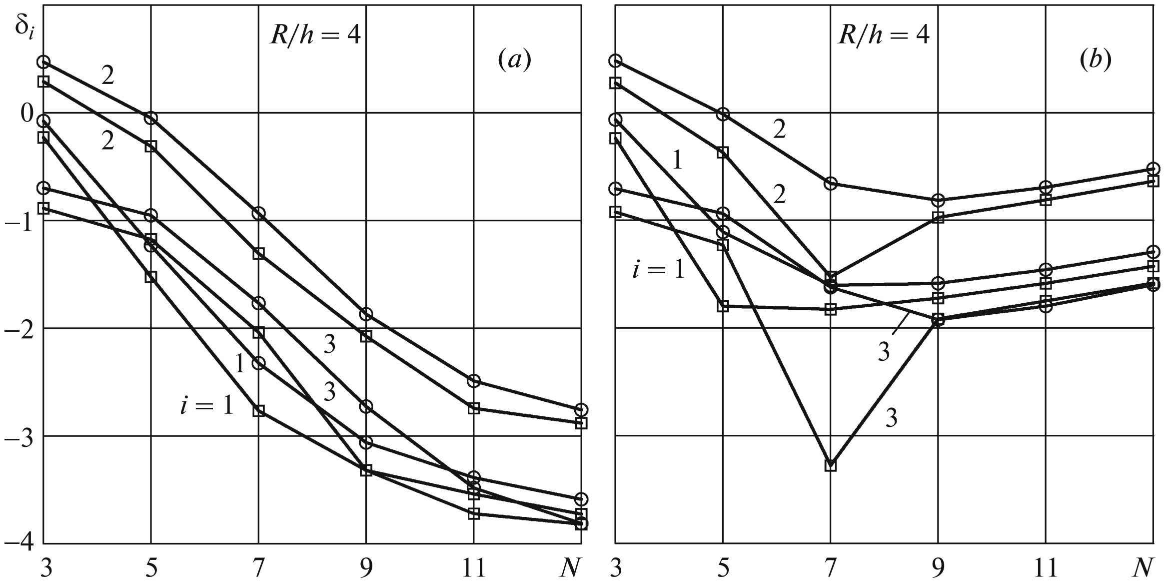

Fig. 6.

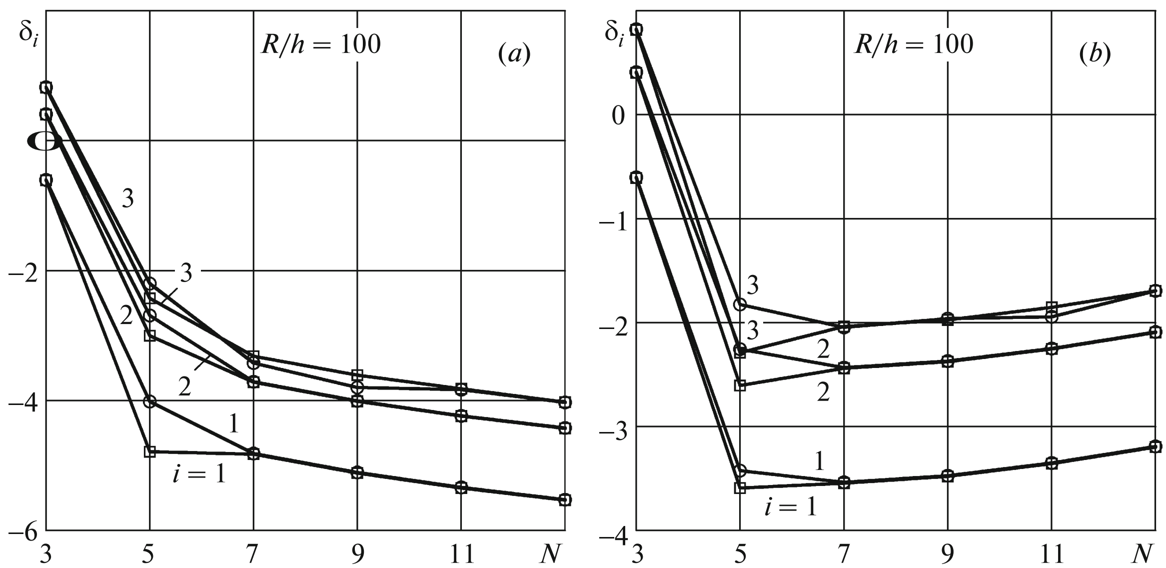

Fig. 7.

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