# Analytical and Finite Element Modeling of Laminated and Functionally Graded Shells

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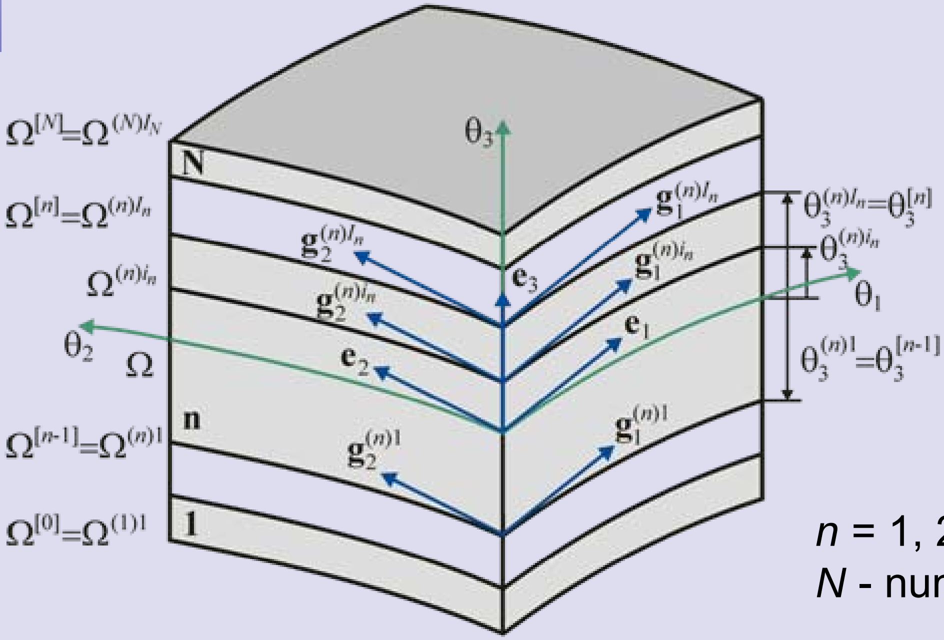
# Kinematic Description of Undeformed Shell




$\Omega^{(n)1}, \Omega^{(n)2}, \ldots, \Omega^{(n)I_n}$ - sampling surfaces (**SaS**)

$\theta_3^{(n)i_n}$ - transverse coordinates of SaS

$\theta_3^{[n-1]}, \theta_3^{[n]}$ - transverse coordinates of interfaces

$$\theta_3^{(n)1} = \theta_3^{[n-1]}, \quad \theta_3^{(n)I_n} = \theta_3^{[n]} \tag{1}$$

$$\theta_3^{(n)m_n} = \frac{1}{2}(\theta_3^{[n-1]} + \theta_3^{[n]}) - \frac{1}{2} h_n \cos\left(\pi \frac{2m_n - 3}{2(I_n - 2)}\right)$$

$n = 1, 2, \ldots, N; \quad i_n = 1, 2, \ldots, I_n; \quad m_n = 2, 3, \ldots, I_n - 1$

$N$ - number of layers; $I_n$ - number of SaS of the $n$th layer

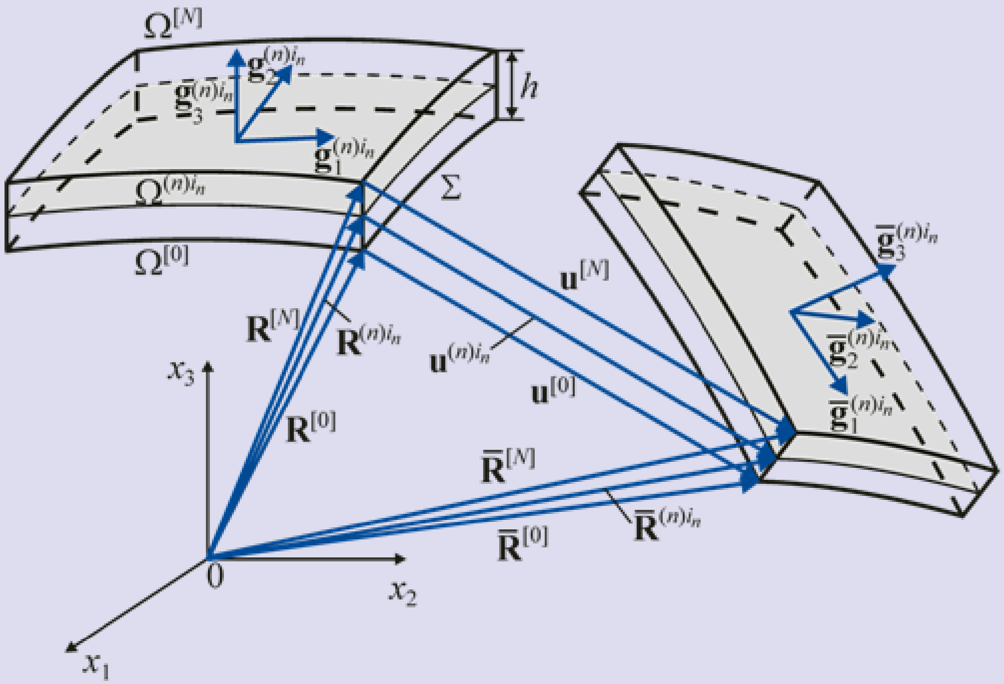
## Position Vectors and Base Vectors of SaS

$$\mathbf{R}^{(n)i_n} = \mathbf{r} + \theta_3^{(n)i_n} \mathbf{e}_3 \tag{2}$$

$$\mathbf{g}_\alpha^{(n)i_n} = \mathbf{R}_{,\alpha}^{(n)i_n} = A_\alpha c_\alpha^{(n)i_n} \mathbf{e}_\alpha, \qquad \mathbf{g}_3^{(n)i_n} = \mathbf{e}_3 \tag{3}$$

$\mathbf{r}(\theta_1, \theta_2)$ – position vector of midsurface $\Omega$ ; $\mathbf{e}_i(\theta_1, \theta_2)$ – orthonormal base vectors; $c_\alpha^{(n)i_n} = 1 + k_\alpha \theta_3^{(n)i_n}$ – components of shifter tensor at SaS; $A_\alpha(\theta_1, \theta_2)$, $k_\alpha(\theta_1, \theta_2)$ – Lamé coefficients and principal curvatures

# Kinematic Description of Deformed Shell




## Position Vectors of Deformed SaS

$$\bar{\mathbf{R}}^{(n)i_n} = \mathbf{R}^{(n)i_n} + \mathbf{u}^{(n)i_n} \quad (4)$$

$$\mathbf{u}^{(n)i_n} = \mathbf{u}(\theta_3^{(n)i_n}) \quad (5)$$

$\mathbf{u}\ (\theta_1, \theta_2, \theta_3)$ – displacement vector

$\mathbf{u}^{(n)i_n}(\theta_1, \theta_2)$ – displacement vectors of SaS

## Base Vectors of Deformed SaS

$$\bar{\mathbf{g}}_\alpha^{(n)i_n} = \bar{\mathbf{R}}_{,\alpha}^{(n)i_n} = \mathbf{g}_\alpha^{(n)i_n} + \mathbf{u}_{,\alpha}^{(n)i_n}, \qquad \bar{\mathbf{g}}_3^{(n)i_n} = \mathbf{e}_3 + \boldsymbol{\beta}^{(n)i_n}, \qquad \boldsymbol{\beta}^{(n)i_n} = \mathbf{u}_{,3}(\theta_3^{(n)i_n}) \quad (6)$$

$\boldsymbol{\beta}^{(n)i_n}(\theta_1, \theta_2)$ – values of derivative of displacement vector at SaS

## Green-Lagrange Strain Tensor at SaS

$$2\varepsilon_{ij}^{(n)i_n} = \frac{1}{A_i A_j c_i^{(n)i_n} c_j^{(n)i_n}} (\bar{\mathbf{g}}_i^{(n)i_n} \cdot \bar{\mathbf{g}}_j^{(n)i_n} - \mathbf{g}_i^{(n)i_n} \cdot \mathbf{g}_j^{(n)i_n}) \qquad (7)$$

## Linearized Strain Tensor at SaS

$$2\varepsilon_{\alpha\beta}^{(n)i_n} = \frac{1}{A_\alpha c_\alpha^{(n)i_n}} \mathbf{u}_{,\alpha}^{(n)i_n} \cdot \mathbf{e}_\beta + \frac{1}{A_\beta c_\beta^{(n)i_n}} \mathbf{u}_{,\beta}^{(n)i_n} \cdot \mathbf{e}_\alpha$$

$$2\varepsilon_{\alpha 3}^{(n)i_n} = \boldsymbol{\beta}^{(n)i_n} \cdot \mathbf{e}_\alpha + \frac{1}{A_\alpha c_\alpha^{(n)i_n}} \mathbf{u}_{,\alpha}^{(n)i_n} \cdot \mathbf{e}_3, \qquad \varepsilon_{33}^{(n)i_n} = \boldsymbol{\beta}^{(n)i_n} \cdot \mathbf{e}_3 \qquad (8)$$

## Displacement Vectors of SaS in Orthonormal Basis $\mathbf{e}_i$:

$$\mathbf{u}^{(n)i_n} = u_i^{(n)i_n} \mathbf{e}_i \qquad (9)$$

## Derivatives of Displacement Vectors in Orthonormal Basis $\mathbf{e}_i$:

$$\frac{1}{A_\alpha}\mathbf{u}^{(n)i_n}_{,\alpha} = \lambda^{(n)i_n}_{i\alpha}\mathbf{e}_i, \qquad \boldsymbol{\beta}^{(n)i_n} = \beta^{(n)i_n}_i \mathbf{e}_i \tag{10}$$

## Strain Parameters of SaS

$$\lambda^{(n)i_n}_{\alpha\alpha} = \frac{1}{A_\alpha}u^{(n)i_n}_{\alpha,\alpha} + B_\alpha u^{(n)i_n}_\beta + k_\alpha u^{(n)i_n}_3, \qquad \lambda^{(n)i_n}_{\beta\alpha} = \frac{1}{A_\alpha}u^{(n)i_n}_{\beta,\alpha} - B_\alpha u^{(n)i_n}_\alpha \quad (\beta \neq \alpha)$$

$$\lambda^{(n)i_n}_{3\alpha} = \frac{1}{A_\alpha}u^{(n)i_n}_{3,\alpha} - k_\alpha u^{(n)i_n}_\alpha, \qquad B_\alpha = \frac{1}{A_\alpha A_\beta}A_{\alpha,\beta} \quad (\beta \neq \alpha) \tag{11}$$

## Linearized Strains of SaS

$$2\varepsilon^{(n)i_n}_{\alpha\beta} = \frac{1}{c^{(n)i_n}_\beta}\lambda^{(n)i_n}_{\alpha\beta} + \frac{1}{c^{(n)i_n}_\alpha}\lambda^{(n)i_n}_{\beta\alpha} \tag{12}$$

$$2\varepsilon^{(n)i_n}_{\alpha 3} = \beta^{(n)i_n}_\alpha + \frac{1}{c^{(n)i_n}_\alpha}\lambda^{(n)i_n}_{3\alpha}, \qquad \varepsilon^{(n)i_n}_{33} = \beta^{(n)i_n}_3$$

***Remark.*** Strains (12) exactly represent all rigid-body shell motions in any convected curvilinear coordinate system. It can be proved through results of Kulikov and Carrera (2008)

## Displacement Distribution in Thickness Direction

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n} u_i^{(n)i_n}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]} \tag{13}$$

$$L^{(n)i_n} = \prod_{j_n \ne i_n} \frac{\theta_3 - \theta_3^{(n)j_n}}{\theta_3^{(n)i_n} - \theta_3^{(n)j_n}} \tag{14}$$

$L^{(n)i_n}(\theta_3)$ – Lagrange polynomials of degree $I_n$ - 1

## Strain Distribution in Thickness Direction

$$\varepsilon_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} \varepsilon_{ij}^{(n)i_n}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]} \tag{15}$$

## Presentation for Derivative of Displacements at SaS

$$\beta_i^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n}) u_i^{(n)j_n}, \qquad M^{(n)j_n} = L_{,3}^{(n)j_n} \tag{16}$$

## Derivatives of Lagrange Polynomials at SaS

$$M^{(n)j_n}(\theta_3^{(n)i_n}) = \frac{1}{\theta_3^{(n)j_n} - \theta_3^{(n)i_n}} \prod_{k_n \neq i_n, j_n} \frac{\theta_3^{(n)i_n} - \theta_3^{(n)k_n}}{\theta_3^{(n)j_n} - \theta_3^{(n)k_n}} \quad \text{for} \quad j_n \neq i_n \tag{17}$$

$$M^{(n)i_n}(\theta_3^{(n)i_n}) = -\sum_{j_n \neq i_n} M^{(n)j_n}(\theta_3^{(n)i_n})$$

# Variational Equation for Analytical Development

$$\delta\Pi = 0 \tag{18}$$

$$\Pi = \frac{1}{2}\iint\limits_{\Omega}\sum_{n}\sum_{i_n} H_{ij}^{(n)i_n}\varepsilon_{ij}^{(n)i_n} A_1 A_2 d\theta_1 d\theta_2 - W \tag{19}$$

$\Pi$ – total potential energy, $W$ – work done by external loads

## Stress Resultants

$$H_{ij}^{(n)i_n} = \int\limits_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} \sigma_{ij}^{(n)} L^{(n)i_n} c_1 c_2 d\theta_3 \tag{20}$$

## Constitutive Equations

$$\sigma_{ij}^{(n)} = C_{ijkl}^{(n)}\varepsilon_{kl}^{(n)}, \qquad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]} \tag{21}$$

$C_{ijkl}^{(n)}$ – elastic constants of the $n$th layer

## Distribution of Elastic Constants in Thickness Direction

$$C_{ijkl}^{(n)} = \sum_{i_n} L^{(n)i_n} C_{ijkl}^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]} \tag{22}$$

$C_{ijkl}^{(n)i_n}$ – values of elastic constants of the $n$th layer on SaS

## Presentations for Stress Resultants

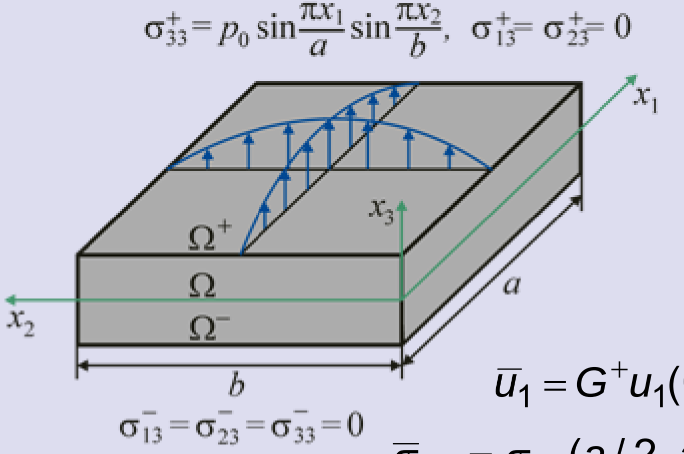
$$H_{ij}^{(n)i_n} = \sum_{j_n,k_n} \Lambda^{(n)i_n j_n k_n} C_{ijkl}^{(n)j_n} \varepsilon_{kl}^{(n)k_n} \tag{23}$$

$$\Lambda^{(n)i_n j_n k_n} = \int_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} L^{(n)i_n} L^{(n)j_n} L^{(n)k_n} c_1 c_2 d\theta_3 \tag{24}$$

$c_\alpha = 1 + k_\alpha \theta_3$ – components of shifter tensor

# Navier-Type Solutions

## 1. FG Rectangular Plate under Sinusoidal Loading




**Analytical solution**

$$u_1^{(n)i_n} = \sum_{r,s} u_{1rs}^{(n)i_n} \cos\frac{r\pi x_1}{a} \sin\frac{s\pi x_2}{b}, \quad u_2^{(n)i_n} = \sum_{r,s} u_{2rs}^{(n)i_n} \sin\frac{r\pi x_1}{a} \cos\frac{s\pi x_2}{b}$$

$$u_3^{(n)i_n} = \sum_{r,s} u_{3rs}^{(n)i_n} \sin\frac{r\pi x_1}{a} \sin\frac{s\pi x_2}{b}$$

**Dimensionless variables in crucial points**

$$\bar{u}_1 = G^{+} u_1(0, a/2, z)/hp_0, \quad \bar{u}_3 = G^{+} u_3(a/2, a/2, z)/hp_0, \quad G^{+} = E^{+}/(2(1+\nu))$$

$$\bar{\sigma}_{11} = \sigma_{11}(a/2,\, a/2,\, z)/p_0, \quad \bar{\sigma}_{13} = \sigma_{13}(0,\, a/2,\, z)/p_0, \quad \bar{\sigma}_{33} = \sigma_{33}(a/2,\, a/2,\, z)/p_0$$

$$E = E^{+} e^{\alpha(z-0.5)}, \quad \alpha = \ln(E^{+}/E^{-}), \quad z = x_3/h, \quad E^{+} = 10^7\ \text{Pa}, \quad \nu = 0.3$$

***Table 1. Results for a square plate with a = b = 1 m , a/h = 3 and $\alpha$ = 0***

| $I_1$ | $-\bar{u}_1(0.5)$ | $\bar{u}_3(0)$ | $\bar{\sigma}_{11}(0.5)$ | $\bar{\sigma}_{13}(0)$ | $\bar{\sigma}_{33}(0)$ |
|---|---|---|---|---|---|
| 3 | 0.3986372494504680 | 1.278878243389591 | 1.999302146854837 | 0.4977436151168003 | 0.4752837277628003 |
| 7 | 0.4358933937131948 | 1.342554953466513 | 2.124032428288413 | 0.7022762666060094 | 0.4943950643281928 |
| 11 | 0.4358933942603120 | 1.342554689543095 | 2.124018410314048 | 0.7023022083223538 | 0.4944039935419638 |
| 15 | 0.4358933942603121 | 1.342554689542491 | 2.124018410193782 | 0.7023022084767580 | 0.4944039936052152 |
| 19 | 0.4358933942603120 | 1.342554689542491 | 2.124018410193780 | 0.7023022084767578 | 0.4944039936052150 |
| Exact | 0.4358933942603120 | 1.342554689542491 | 2.124018410193781 | 0.7023022084767578 | 0.4944039936052149 |

Exact results have been obtained by authors using Vlasov's closed-form solution (1957)

**Table 2. Results for a FG square plate with a = b = 1 m , a/h = 3 and $\alpha$ = 0.1**

| $I_1$ | $-\bar{u}_1(0.5)$ | $\bar{u}_3(0)$ | $\bar{\sigma}_{11}(0.5)$ | $\bar{\sigma}_{13}(0)$ | $\bar{\sigma}_{33}(0)$ |
|---|---|---|---|---|---|
| 3 | 0.4158336502652652 | 1.347977241257294 | 2.073239906807475 | 0.4983226799021384 | 0.4596565434470269 |
| 7 | 0.4536977979133792 | 1.414636043682728 | 2.193265858989844 | 0.7020700339720654 | 0.4877173446168515 |
| 11 | 0.4536977984142576 | 1.414635771310962 | 2.193270258459039 | 0.7020957676355946 | 0.4877129579452970 |
| 15 | 0.4536977984142576 | 1.414635771310368 | 2.193270258650021 | 0.7020957677904856 | 0.4877129578319663 |
| 19 | 0.4536977984142575 | 1.414635771310368 | 2.193270258650021 | 0.7020957677904854 | 0.4877129578319657 |
| Exact | | 1.41464 | | | |

Exact results have been obtained by Kashtalyan (2004)

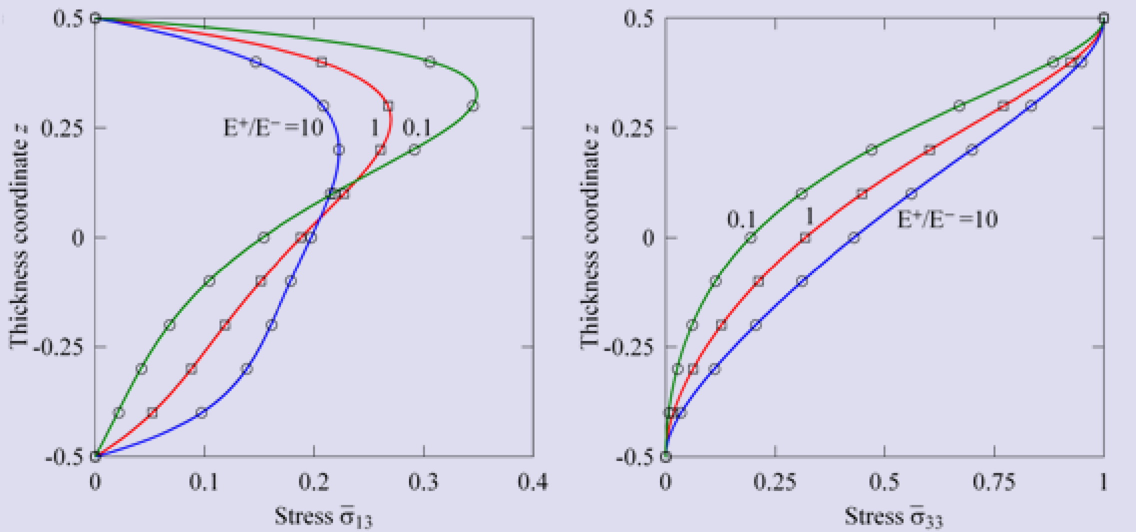
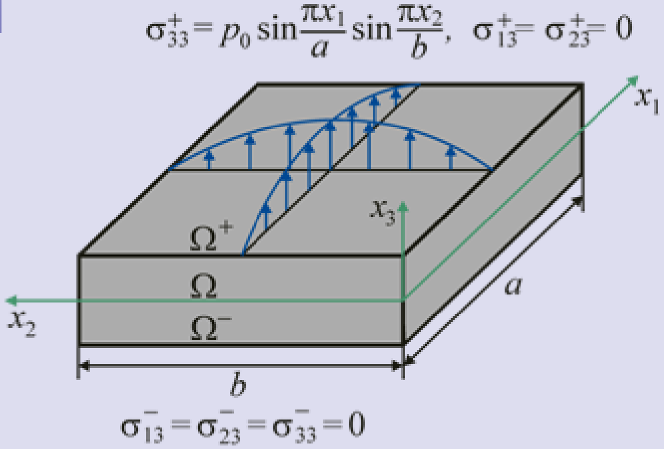



***Figure 1. Through-thickness distributions of transverse stresses of FG plate with a/h = 1: present analysis (–) for $I_1$ = 11 and Vlasov's closed-form solution (□), and Kashtalyan's closed-form solution (○)***

# 2. Two-Phase FG Rectangular Plate under Sinusoidal Loading




**Analytical solution**

$$u_1^{(n)i_n} = \sum_{r,s} u_{1rs}^{(n)i_n} \cos\frac{r\pi x_1}{a} \sin\frac{s\pi x_2}{b}, \quad u_2^{(n)i_n} = \sum_{r,s} u_{2rs}^{(n)i_n} \sin\frac{r\pi x_1}{a} \cos\frac{s\pi x_2}{b}$$

$$u_3^{(n)i_n} = \sum_{r,s} u_{3rs}^{(n)i_n} \sin\frac{r\pi x_1}{a} \sin\frac{s\pi x_2}{b}$$

**Mori-Tanaka method is used for evaluating material properties of the Metal/Ceramic plate with**

$$V_{\mathrm{m}} = 1 - V_{\mathrm{c}}, \quad V_{\mathrm{c}} = V_{\mathrm{c}}^{-} + (V_{\mathrm{c}}^{+} - V_{\mathrm{c}}^{-})(0.5 + z)^{\gamma}, \quad z = x_3 / h, \quad V_c^{-} = 0, \quad V_c^{+} = 0.5, \quad \gamma = 2$$

where $V_c^{-}$, $V_c^{+}$ – volume fractions of the ceramic phase on bottom and top surfaces

**Dimensionless variables in crucial points**

$$\bar{u}_3 = 100 h^3 E_{\mathrm{m}} u_3(a/2, a/2, z) / a^4 p_0, \quad \bar{\sigma}_{11} = 10 h^2 \sigma_{11}(a/2, a/2, z) / a^2 p_0$$

$$\bar{\sigma}_{12} = 10 h^2 \sigma_{12}(0, 0, z) / a^2 p_0, \quad \bar{\sigma}_{13} = 10 h \sigma_{13}(0, a/2, z) / a p_0$$

$$\bar{\sigma}_{33} = \sigma_{33}(a/2, a/2, z) / p_0, \quad a = b = 1\,\mathrm{m}$$

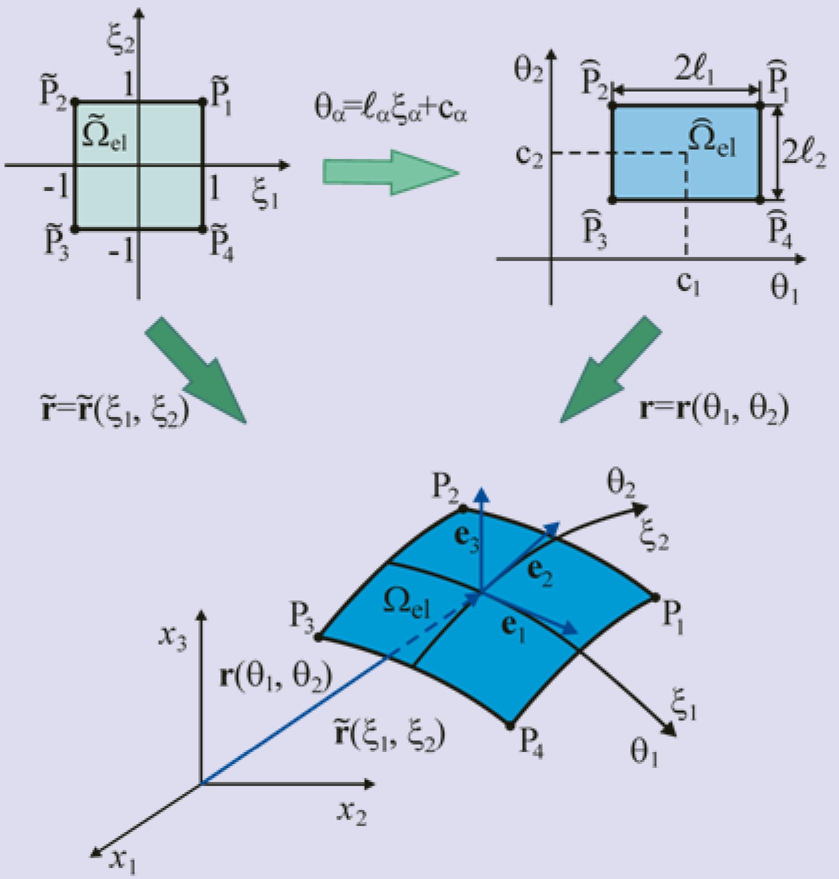
**Table 3. Results for a Metal/Ceramic square plate with a/h = 5**

| $I_1$ | $\bar{u}_3(0.5)$ | $\bar{\sigma}_{11}(0.5)$ | $\bar{\sigma}_{12}(0.5)$ | $\bar{\sigma}_{13}(0)$ | $\bar{\sigma}_{33}(0.25)$ |
|---|---|---|---|---|---|
| 3 | 2.491469749603327 | 2.501382580630834 | -1.396928304286311 | 1.590619403769780 | 0.7124999023164764 |
| 7 | 2.555878887142974 | 2.755727774481453 | -1.559557127193362 | 2.310138127165216 | 0.8099883632143655 |
| 11 | 2.555881463601872 | 2.756212114487591 | -1.559952113334130 | 2.310019719768843 | 0.8100152119426805 |
| 15 | 2.555881465062731 | 2.756213284948613 | -1.559953146848270 | 2.310019478367794 | 0.8100148488194952 |
| 19 | 2.555881465064410 | 2.756213287523302 | -1.559953149637165 | 2.310019478154247 | 0.8100148486775266 |
| 23 | 2.555881465064128 | 2.756213287528492 | -1.559953149644632 | 2.310019478154246 | 0.8100148486802695 |
| Batra | 2.5559 | 2.7562 | -1.5600 | 2.3100 | 0.8100 |

**Table 4. Results for a Metal/Ceramic square plate with a/h = 10**

| $I_1$ | $\bar{u}_3(0.5)$ | $\bar{\sigma}_{11}(0.5)$ | $\bar{\sigma}_{12}(0.5)$ | $\bar{\sigma}_{13}(0)$ | $\bar{\sigma}_{33}(0.25)$ |
|---|---|---|---|---|---|
| 3 | 2.202396920585996 | 2.412841121703996 | -1.420413278358082 | 1.601325513211140 | 0.7367643589818127 |
| 7 | 2.214798606780433 | 2.641956852342622 | -1.552565106480717 | 2.323954032297057 | 0.8117297257693006 |
| 11 | 2.214801182167397 | 2.642395937995539 | -1.552891720288181 | 2.323921893370359 | 0.8123146819945828 |
| 15 | 2.214801183896660 | 2.642396888131435 | -1.552892592322280 | 2.323921660091405 | 0.8123134893082843 |
| 19 | 2.214801183898813 | 2.642396890147046 | -1.552892594687380 | 2.323921659883698 | 0.8123134883311503 |
| 23 | 2.214801183898711 | 2.642396890150937 | -1.552892594693696 | 2.323921659883744 | 0.8123134883405911 |
| Batra | 2.2148 | 2.6424 | -1.5529 | 2.3239 | 0.8123 |

# Exact Geometry Solid-Shell Element Formulation




*Biunit square in ($\xi_1$, $\xi_2$)-space mapped into the exact geometry four-node shell element in ($x_1$, $x_2$, $x_3$)-space*

## Displacement Interpolation

$$u_i^{(n)i_n} = \sum_{r=1}^{4} N_r u_{ir}^{(n)i_n}$$

$$u_{ir}^{(n)i_n} = u_i^{(n)i_n}\left(\tilde{P}_r\right) \qquad (25)$$

## Assumed Strain Interpolation

$$\varepsilon_{ij}^{(n)i_n} = \sum_{r=1}^{4} N_r \varepsilon_{ijr}^{(n)i_n}$$

$$\varepsilon_{ijr}^{(n)i_n} = \varepsilon_{ij}^{(n)i_n}\left(\tilde{P}_r\right) \qquad (26)$$

$N_r(\xi_1, \xi_2)$ - bilinear shape functions

$\xi_\alpha = (\theta_\alpha - c_\alpha)/\ell_\alpha$ - normalized surface coordinates

## Assumed Stress Resultant Interpolation

$$H_{11}^{(n)i_n} = \Phi_1^{(n)i_n} + \xi_2 \Phi_7^{(n)i_n}, \qquad H_{22}^{(n)i_n} = \Phi_2^{(n)i_n} + \xi_1 \Phi_8^{(n)i_n}$$

$$H_{33}^{(n)i_n} = \Phi_3^{(n)i_n} + \xi_1 \Phi_9^{(n)i_n} + \xi_2 \Phi_{10}^{(n)i_n}, \qquad H_{12}^{(n)i_n} = \Phi_4^{(n)i_n} \qquad (27)$$

$$H_{13}^{(n)i_n} = \Phi_5^{(n)i_n} + \xi_2 \Phi_{11}^{(n)i_n}, \qquad H_{23}^{(n)i_n} = \Phi_6^{(n)i_n} + \xi_1 \Phi_{12}^{(n)i_n}$$

## Assumed Displacement-Independent Strain Interpolation

$$e_{11}^{(n)i_n} = \Psi_1^{(n)i_n} + \xi_2 \Psi_7^{(n)i_n}, \qquad e_{22}^{(n)i_n} = \Psi_2^{(n)i_n} + \xi_1 \Psi_8^{(n)i_n}$$

$$e_{33}^{(n)i_n} = \Psi_3^{(n)i_n} + \xi_1 \Psi_9^{(n)i_n} + \xi_2 \Psi_{10}^{(n)i_n}, \qquad e_{12}^{(n)i_n} = \Psi_4^{(n)i_n} \qquad (28)$$

$$e_{13}^{(n)i_n} = \Psi_5^{(n)i_n} + \xi_2 \Psi_{11}^{(n)i_n}, \qquad e_{23}^{(n)i_n} = \Psi_6^{(n)i_n} + \xi_1 \Psi_{12}^{(n)i_n}$$

## Hu-Washizu Mixed Variational Equation

$$\iint_{\Omega}\sum_{n}\sum_{i_n}\left[\delta\left(\mathbf{e}^{(n)i_n}\right)^{\mathrm{T}}\left(\mathbf{H}^{(n)i_n}-\sum_{j_n,k_n}\Lambda^{(n)i_n j_n k_n}\mathbf{C}^{(n)j_n}\mathbf{e}^{(n)k_n}\right)\right.$$

$$\left.+\,\delta\left(\mathbf{H}^{(n)i_n}\right)^{\mathrm{T}}\left(\mathbf{e}^{(n)i_n}-\boldsymbol{\varepsilon}^{(n)i_n}\right)-\delta\left(\boldsymbol{\varepsilon}^{(n)i_n}\right)^{\mathrm{T}}\mathbf{H}^{(n)i_n}\right]A_1A_2\mathrm{d}\theta_1\mathrm{d}\theta_2 \tag{29}$$

$$+\iint_{\Omega}\left(c_1^+c_2^+p_i^+\delta u_i^+-c_1^-c_2^-p_i^-\delta u_i^-\right)A_1A_2\mathrm{d}\theta_1\mathrm{d}\theta_2+\delta W_{\Sigma}=0$$

$W_{\Sigma}$ – work done by external loads applied to edge surface $\Sigma$

$$\boldsymbol{\varepsilon}^{(n)i_n}=\left[\varepsilon_{11}^{(n)i_n}\ \ \varepsilon_{22}^{(n)i_n}\ \ \varepsilon_{33}^{(n)i_n}\ \ 2\varepsilon_{12}^{(n)i_n}\ \ 2\varepsilon_{13}^{(n)i_n}\ \ 2\varepsilon_{23}^{(n)i_n}\right]^{\mathrm{T}}$$

$$\mathbf{e}^{(n)i_n}=\left[e_{11}^{(n)i_n}\ \ e_{22}^{(n)i_n}\ \ e_{33}^{(n)i_n}\ \ 2e_{12}^{(n)i_n}\ \ 2e_{13}^{(n)i_n}\ \ 2e_{23}^{(n)i_n}\right]^{\mathrm{T}} \tag{30}$$

$$\mathbf{H}^{(n)i_n}=\left[H_{11}^{(n)i_n}\ \ H_{22}^{(n)i_n}\ \ H_{33}^{(n)i_n}\ \ H_{12}^{(n)i_n}\ \ H_{13}^{(n)i_n}\ \ H_{23}^{(n)i_n}\right]^{\mathrm{T}}$$

$$\mathbf{C}^{(n)i_n} = \begin{bmatrix} C_{1111}^{(n)i_n} & C_{1122}^{(n)i_n} & C_{1133}^{(n)i_n} & C_{1112}^{(n)i_n} & 0 & 0 \\ C_{2211}^{(n)i_n} & C_{2222}^{(n)i_n} & C_{2233}^{(n)i_n} & C_{2212}^{(n)i_n} & 0 & 0 \\ C_{3311}^{(n)i_n} & C_{3322}^{(n)i_n} & C_{3333}^{(n)i_n} & C_{3312}^{(n)i_n} & 0 & 0 \\ C_{1211}^{(n)i_n} & C_{1222}^{(n)i_n} & C_{1233}^{(n)i_n} & C_{1212}^{(n)i_n} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313}^{(n)i_n} & C_{1323}^{(n)i_n} \\ 0 & 0 & 0 & 0 & C_{2313}^{(n)i_n} & C_{2323}^{(n)i_n} \end{bmatrix} \quad (31)$$

## Finite Element Equations

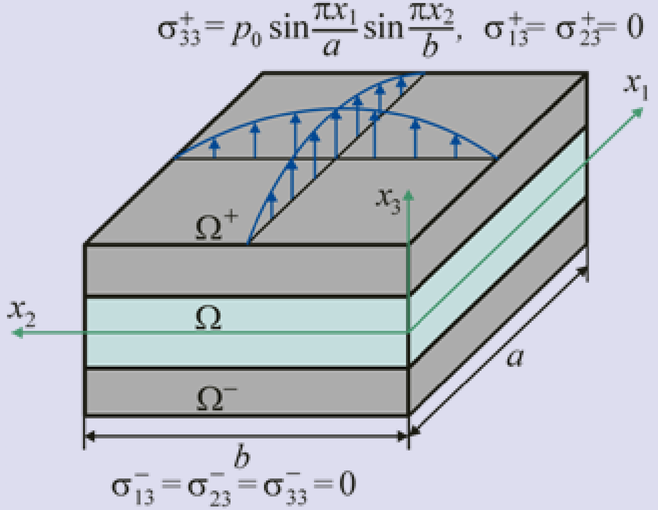
$$\mathbf{KU} = \mathbf{F} \quad (32)$$

**K** – element stiffness matrix of order $12N_{\mathrm{SaS}} \times 12N_{\mathrm{SaS}}$, where $N_{\mathrm{SaS}} = \sum_n I_n - N + 1$

$\mathbf{U} = \left[\mathbf{U}_1^{\mathrm{T}}\ \mathbf{U}_2^{\mathrm{T}}\ \mathbf{U}_3^{\mathrm{T}}\ \mathbf{U}_4^{\mathrm{T}}\right]^{\mathrm{T}}$ – element displacement vector, $\mathbf{U}_r$ – nodal displacement vector ($r$ = 1, 2, 3, 4)

# 3D Finite Element Solutions

## 1. Three-Layer Rectangular Plate under Sinusoidal Loading




**Dimensionless variables in crucial points**

$$\bar{u}_3 = 100 E_T h^3 u_3(a/2,\, b/2,\, z)/a^4 p_0, \quad \bar{\sigma}_{11} = h^2 \sigma_{11}(a/2,\, b/2,\, z)/a^2 p_0$$

$$\bar{\sigma}_{22} = 10 h^2 \sigma_{22}(a/2,\, b/2,\, z)/a^2 p_0, \quad \bar{\sigma}_{12} = 10 h^2 \sigma_{12}(0,\, 0,\, z)/a^2 p_0$$

$$\bar{\sigma}_{13} = 10 h \sigma_{13}(0,\, b/2,\, z)/a p_0, \quad \bar{\sigma}_{23} = 10 h \sigma_{23}(a/2,\, 0,\, z)/a p_0$$

$$\bar{\sigma}_{33} = \sigma_{33}(a/2,\, b/2,\, z)/p_0, \quad z = x_3/h$$

**Lamination scheme:** $[0/90/0]$ and $\left[\frac{1}{3}h/\frac{1}{3}h/\frac{1}{3}h\right]$

$$E_L = 25E_T, \quad G_{LT} = 0.5E_T, \quad G_{TT} = 0.2E_T, \quad E_T = 10^6, \quad \nu_{LT} = \nu_{TT} = 0.25, \quad b = 3a, \quad a = 1$$

***Table 5. Results for a three-layer rectangular plate with a/h = 4 using a regular 64×64 mesh***

| Formulation | $\bar{u}_3(0)$ | $\bar{\sigma}_{11}(-0.5)$ | $\bar{\sigma}_{11}(0.5)$ | $\bar{\sigma}_{22}(-1/6)$ | $\bar{\sigma}_{22}(1/6)$ | $10\bar{\sigma}_{12}(-0.5)$ | $10\bar{\sigma}_{12}(0.5)$ | $\bar{\sigma}_{13}(0)$ | $10\bar{\sigma}_{23}(0)$ | $\bar{\sigma}_{33}(0.5)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $I_n = 3$ | 2.79852 | -1.08234 | 1.12692 | -1.18341 | 1.07900 | 2.77774 | -2.65861 | -3.47183 | -2.93050 | 1.05038 |
| $I_n = 4$ | 2.82122 | -1.09941 | 1.14453 | -1.19290 | 1.08778 | 2.80636 | -2.68763 | -3.51076 | -3.33734 | 1.02052 |
| $I_n = 5$ | 2.82133 | -1.09920 | 1.14430 | -1.19290 | 1.08778 | 2.80654 | -2.68782 | -3.51073 | -3.33748 | 1.00108 |
| $I_n = 6$ | 2.82134 | -1.09919 | 1.14429 | -1.19291 | 1.08778 | 2.80655 | -2.68783 | -3.51074 | -3.33658 | 1.00014 |
| $I_n = 7$ | 2.82134 | -1.09919 | 1.14429 | -1.19291 | 1.08778 | 2.80655 | -2.68783 | -3.51074 | -3.33658 | 0.99987 |
| Exact SaS | 2.8211 | -1.0992 | 1.1443 | -1.1929 | 1.0878 | 2.8065 | -2.6878 | -3.5108 | -3.3365 | 1.0000 |
| Pagano | 2.82 | -1.10 | 1.14 | -1.19 | 1.09 | 2.81 | -2.69 | -3.51 | -3.34 | 1.00 |

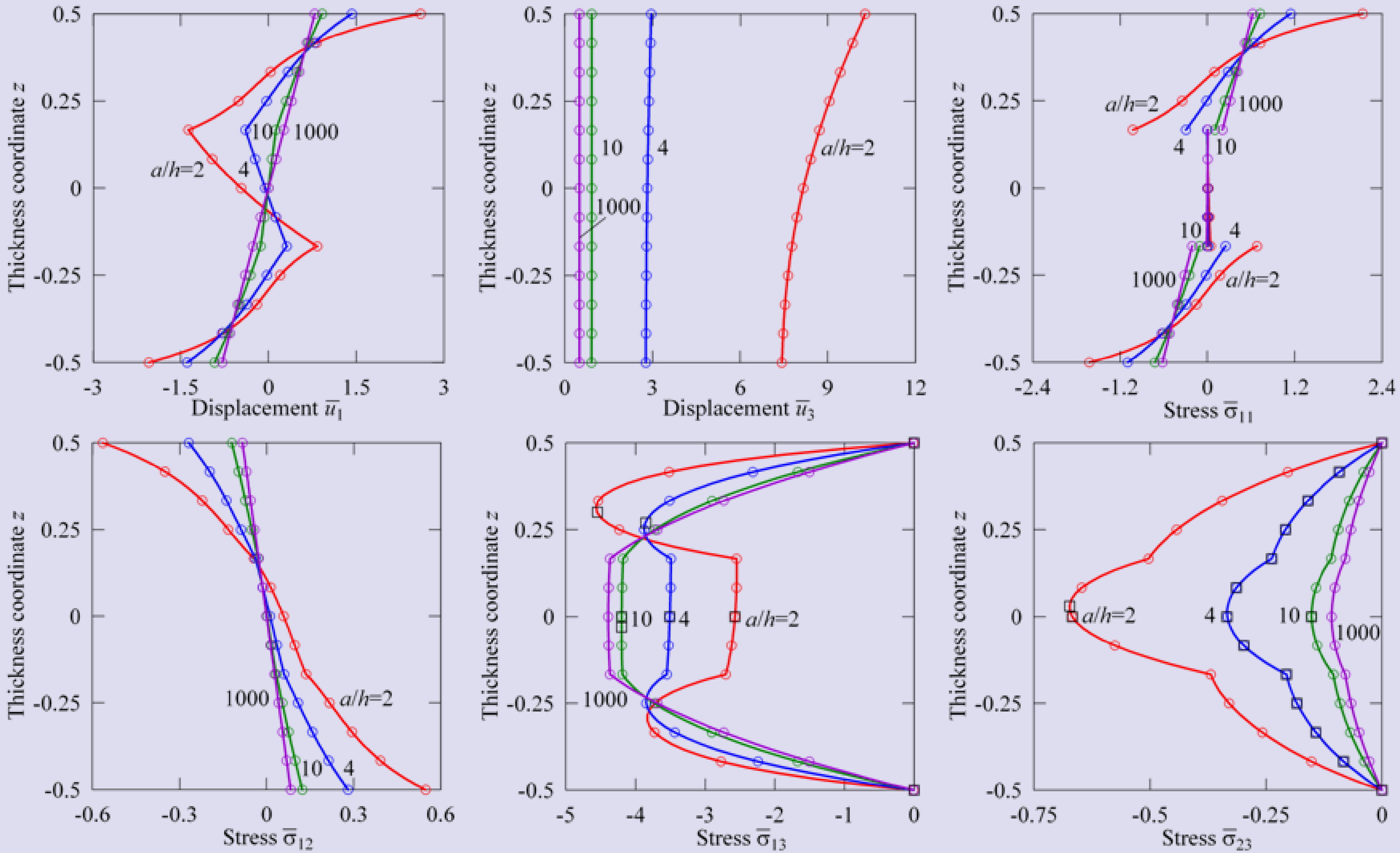
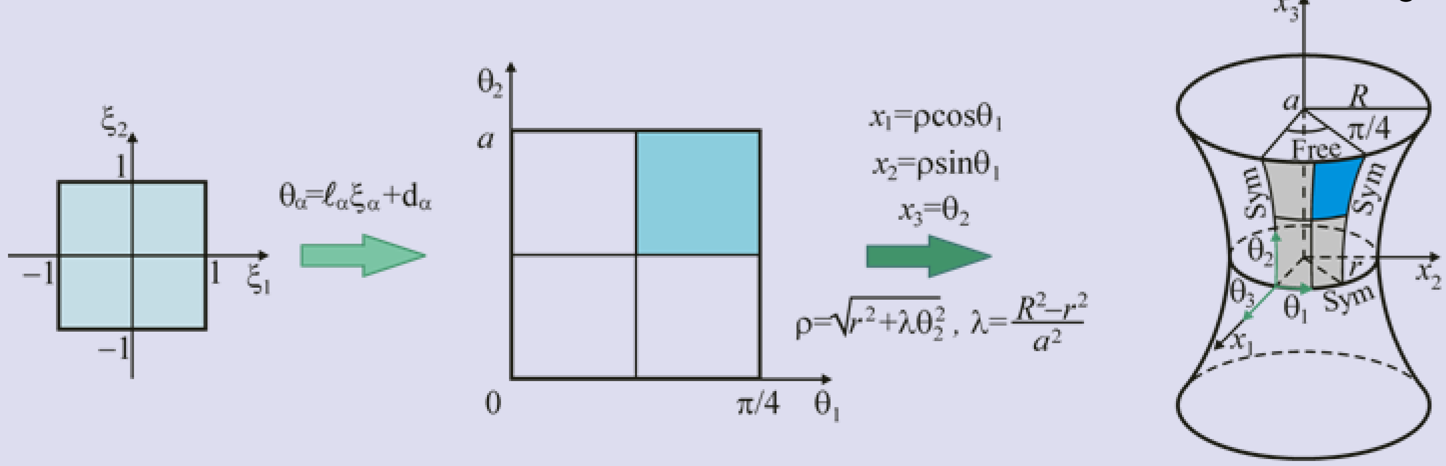



***Figure 2. Through-thickness distributions of displacements and stresses for three-layer rectangular plate for $I_n$ = 7 using a regular 64×64 mesh; exact SaS solution (○) and Pagano's closed form solution (□)***

## 2. Hyperbolic Composite Shell under Inner Pressure $p_3^- = -p_0 \cos 4\theta_2$




Single-layer graphite/epoxy hyperbolic shell with $r = 7.5$, $R = 15$, $a = 20$, $E_1 = 40E$, $E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} = 0.6E$, $E = 10^6$ and $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$ is modeled by $64 \times 64$ mesh

$$\bar\sigma_{13} = 10a\sigma_{13}(0,0,z)/hp_0, \quad \bar\sigma_{23} = 10a\sigma_{23}(\pi/8,a,z)/hp_0, \quad \bar\sigma_{33} = \sigma_{33}(0,a,z)/p_0, \quad z = \theta_3/h$$

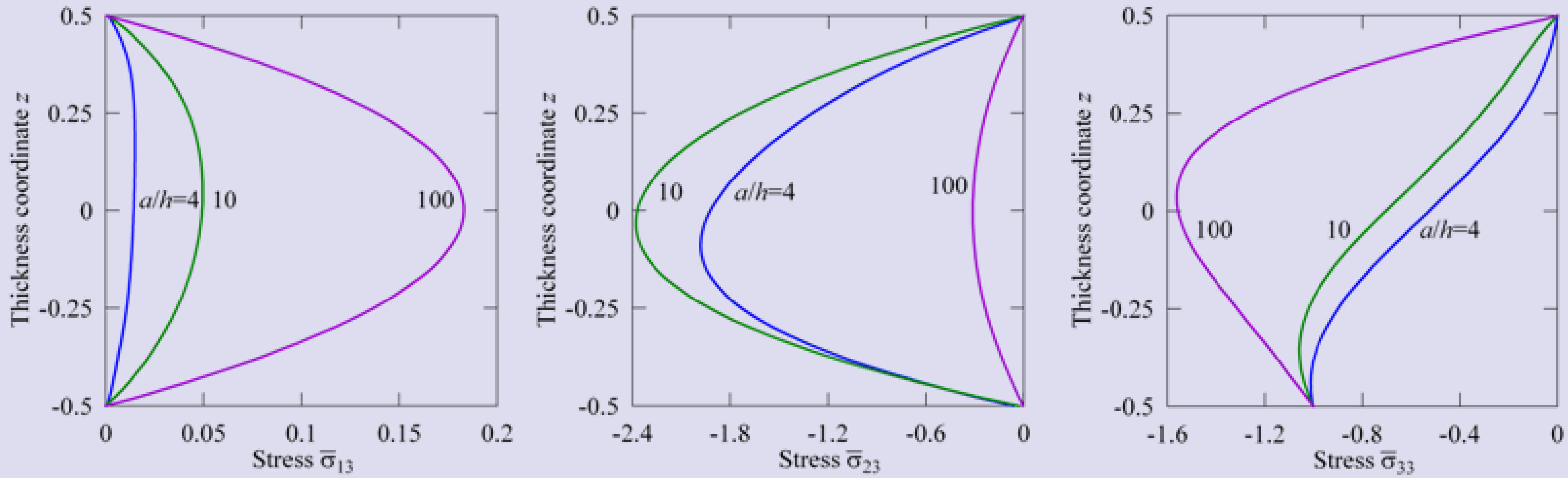



*Figure 3. Distribution of stresses through the shell thickness for $l_1 = 7$*

# Conclusions

- An efficient concept of sampling surfaces inside the shell body has been proposed. This concept permits the use of 3D constitutive equations and leads for the large number of sampling surfaces to numerical solutions for layered composite shells which asymptotically approach the 3D solutions of elasticity
- A robust exact geometry four-node solid-shell element has been developed which allows the solution of 3D elasticity problems for thick and thin shells of arbitrary geometry using only displacement degrees of freedom

## Thanks for your attention!