STRONG AND WEAK SAMPLING SURFACES FORMULATIONS FOR THREE-DIMENSIONAL STRESS ANALYSIS OF PIEZOELECTRIC PLATES

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Abstract. This paper focuses on implementation of the sampling surfaces (SaS) method [1] for the three-dimensional (3D) stress analysis of layered piezoelectric plates. The SaS formulation is based on choosing inside the layers the arbitrary number of not equally spaced SaS parallel to the middle surface in order to introduce the displacements and electric potentials of these surfaces as basic plate unknowns. Such choice of unknowns permits the presentation of the proposed piezoelectric plate formulation in a very compact form. The SaS are located inside each layer at Chebyshev polynomial nodes that improves the convergence of the SaS method significantly. Therefore, the SaS formulation can be applied efficiently to analytical solutions for layered piezoelectric plates, which asymptotically approach the 3D exact solutions of electroelasticity as the number of SaS tends to infinity. The strong SaS formulation is based on integrating the equilibrium equations of piezoelectricity, whereas the weak SaS formulation is based on a variational approach proposed earlier by the author [2].

1 VARIATIONAL SAS FORMULATION

Consider a layered piezoelectric plate of the thickness h. Let the middle surface Ω be described by Cartesian coordinates x_1 and x_2. The coordinate x_3 is oriented in the thickness direction. According to the SaS concept, we choose inside the nth layer I_n SaS $\Omega^{(n)1}, \Omega^{(n)2}, ..., \Omega^{(n)I_n}$ parallel to the middle surface. The transverse coordinates of SaS of the nth layer located at Chebyshev polynomial nodes are written as

$$x_3^{(n)i_n} = \frac{1}{2}(x_3^{[n-1]} + x_3^{[n]}) - \frac{1}{2}h_n \cos\left(\pi \frac{2i_n - 1}{2I_n}\right), \quad (1)$$

where $x_3^{[n-1]}$ and $x_3^{[n]}$ are the transverse coordinates of interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$; $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the nth layer; the index $n = 1, 2, ..., N$ identifies the belonging of any quantity to the nth layer, where N is the number of layers; the index $i_n = 1, 2, ..., I_n$ identifies the belonging of any quantity to the SaS of the nth layer.

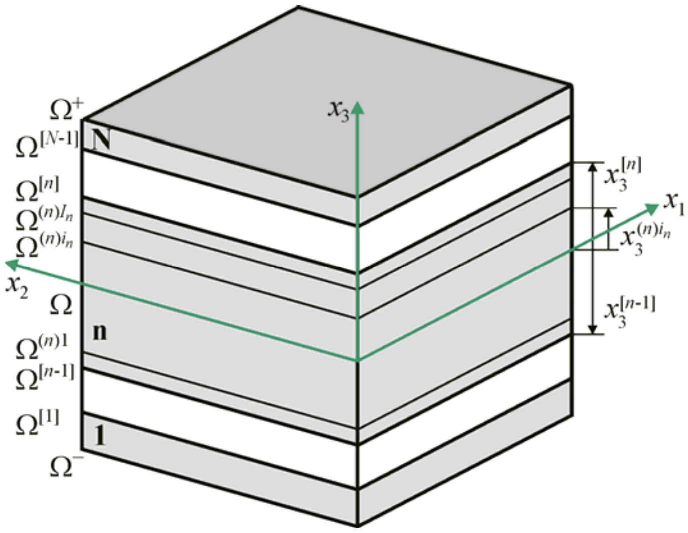

Figure 1: Geometry of the layered piezoelectric plate

The through-the-thickness SaS approximations can be expressed as

$$[u_i^{(n)},\, \varepsilon_{ij}^{(n)},\, \sigma_{ij}^{(n)},\, \varphi^{(n)},\, E_i^{(n)},\, D_i^{(n)}] = \sum_{i_n} L^{(n)i_n} [u_i^{(n)i_n},\, \varepsilon_{ij}^{(n)i_n},\, \sigma_{ij}^{(n)i_n},\, \varphi^{(n)i_n},\, E_i^{(n)i_n},\, D_i^{(n)i_n}], \quad (2)$$

where $u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}, D_i^{(n)}$ are the displacements, strains, stresses, electric potential, electric field and electric displacements of the nth layer; $u_i^{(n)i_n}, \varepsilon_{ij}^{(n)i_n}, \sigma_{ij}^{(n)i_n}, \varphi^{(n)i_n}, E_i^{(n)i_n}, D_i^{(n)i_n}$ are the displacements, strains, stresses, electric potential, electric field and electric displacements of SaS of the nth layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(x_3)$ are the Lagrange basis polynomials of degree $I_n - 1$ corresponding to the nth layer:

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}} \quad (i_n, j_n = 1, 2, \ldots, I_n). \quad (3)$$

The variational SaS formulation for the laminated piezoelectric plate is based on a variational equation

$$\delta \iint_{\Omega} \sum_n \int_{x_3^{[n-1]}}^{x_3^{[n]}} \frac{1}{2}\left(\sigma_{ij}^{(n)}\varepsilon_{ij}^{(n)} - D_i^{(n)}E_i^{(n)}\right) dx_1 dx_2 dx_3 = \delta W, \quad (4)$$

where W is the work done by external electromechanical loads. Here, the summation on repeated Latin indices is implied.

2 STRONG SAS FORMULATION

For simplicity, we consider the case of linear piezoelectric materials given by

$$\sigma_{ij}^{(n)} = C_{ijkl}^{(n)}\varepsilon_{kl}^{(n)} - e_{kij}^{(n)}E_k^{(n)}, \quad D_i^{(n)} = e_{ikl}^{(n)}\varepsilon_{kl}^{(n)} + \in_{ik}^{(n)} E_k^{(n)}, \quad (5)$$

where $C_{ijkl}^{(n)}$, $e_{kij}^{(n)}$ and $\in_{ik}^{(n)}$ are the elastic, piezoelectric and dielectric constants of the nth layer.

The equilibrium equations and charge equation of electrostatics in the absence of body forces and free charges can be written as

$$\sigma_{ij,j}^{(n)} = 0, \tag{6}$$

$$D_{i,i}^{(n)} = 0, \tag{7}$$

where the symbol $(\ldots)_{,i}$ stands for the partial derivatives with respect to coordinates x_i.

The boundary conditions on bottom and top surfaces are defined as

$$u_i^{(1)}(-h/2) = w_i^- \ \text{ or } \ \sigma_{i3}^{(1)}(-h/2) = p_i^-, \quad \varphi^{(1)}(-h/2) = \Phi^- \ \text{ or } \ D_3^{(1)}(-h/2) = Q^-, \tag{8}$$

$$u_i^{(N)}(h/2) = w_i^+ \ \text{ or } \ \sigma_{i3}^{(N)}(h/2) = p_i^+, \quad \varphi^{(N)}(h/2) = \Phi^+ \ \text{ or } \ D_3^{(N)}(h/2) = Q^+, \tag{9}$$

where w_i^-, p_i^-, Φ^-, Q^- and w_i^+, p_i^+, Φ^+, Q^+ are the prescribed displacements, surface tractions, electric potentials and electric charges at the bottom and top surfaces.

The continuity conditions at interfaces are

$$\begin{aligned} u_i^{(m)}(x_3^{[m]}) = u_i^{(m+1)}(x_3^{[m]}), \quad \sigma_{i3}^{(m)}(x_3^{[m]}) = \sigma_{i3}^{(m+1)}(x_3^{[m]}), \\ \varphi^{(m)}(x_3^{[m]}) = \varphi^{(m+1)}(x_3^{[m]}), \quad D_3^{(m)}(x_3^{[m]}) = D_3^{(m+1)}(x_3^{[m]}), \end{aligned} \tag{10}$$

where the index $m = 1, 2, ..., N-1$ identifies the belonging of any quantity to the interface $\Omega^{[m]}$.

Satisfying the equilibrium equations and charge equation (6) and (7) at inner points $x_3^{(n)m_n}$ inside the layers, the following differential equations are obtained:

$$\sigma_{i1,1}^{(n)m_n} + \sigma_{i2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n})\sigma_{i3}^{(n)i_n} = 0, \tag{11}$$

$$D_{1,1}^{(n)m_n} + D_{2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n}) D_3^{(n)i_n} = 0, \tag{12}$$

where $M^{(n)i_n} = L_{,3}^{(n)i_n}$ are the derivatives of the Lagrange basis polynomials whose values at SaS $\Omega^{(n)m_n}$ are evaluated in papers [1, 2] and $m_n = 2, 3, ..., I_n - 1$.

Next, we satisfy the boundary conditions

$$\begin{aligned} \sum_{i_1} L^{(1)i_1}(-h/2) u_i^{(1)i_1} = w_i^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2)\sigma_{i3}^{(1)i_1} = p_i^-, \\ \sum_{i_1} L^{(1)i_1}(-h/2)\varphi^{(1)i_1} = \Phi^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2) D_3^{(1)i_1} = Q^-, \end{aligned} \tag{13}$$

$$\begin{aligned} \sum_{i_N} L^{(N)i_N}(h/2) u_i^{(N)i_N} = w_i^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2)\sigma_{i3}^{(N)i_N} = p_i^+, \\ \sum_{i_N} L^{(N)i_N}(h/2)\varphi^{(1)i_N} = \Phi^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2) D_3^{(N)i_N} = Q^+ \end{aligned} \tag{14}$$

and the continuity conditions that result in

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) u_i^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) u_i^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) \sigma_{i3}^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) \sigma_{i3}^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) \varphi^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) \varphi^{(m+1)i_{m+1}},$$
$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) D_3^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) D_3^{(m+1)i_{m+1}}. \quad (15)$$

Thus, the proposed strong SaS formulation deals with $4(I_1 + I_2 + ... + I_N)$ governing equations (11)-(15) for finding the same number of SaS displacements $u_i^{(n)i_n}$ and SaS electric potentials $\varphi^{(n)i_n}$. These differential and algebraic equations have to be solved to describe the response of the layered piezoelectric plate.

3 NUMERICAL EXAMPLE

Consider a symmetric three-layer piezoelectric square plate with the stacking sequence [0/90/0] and thicknesses $h_1 = h_2 = h_3 = h/3$ composed of PVDF whose material properties are given in [2, 3]. The plate is subjected to the sinusoidally distributed transverse load defined as

$$p_3^- = 0, \quad p_3^+ = p_0 \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{b}.$$

The bottom and top surfaces are assumed to be electrically grounded. To compare the results derived with the exact solution of Heyliger [3], we accept $h = 0.01\,\mathrm{m}$ and $p_0 = 3\,\mathrm{Pa}$.

Figure 2 shows the through-thickness distributions of transverse shear stresses, electric potential and electric displacement for different values of the slenderness ratio a/h taking seven SaS for each layer. These results demonstrate convincingly the high potential of the strong SaS formulation because the boundary conditions on bottom and top surfaces for the transverse stresses and the continuity conditions at interfaces for the transverse stresses and electric displacement are satisfied exactly. Note that in a variational SaS formulation [2] the boundary conditions on outer surfaces and the continuity conditions at interfaces are satisfied approximately.

Consider next the same symmetric three-layer square plate subjected to electric loading

$$\Phi^- = 0, \quad \Phi^+ = \Phi_0 \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{b}.$$

The bottom and top surfaces of the plate are assumed to be traction free. The obtained results are compared with Heyliger's exact solution [3] in the case of choosing $h = 0.01\,\mathrm{m}$ and $\Phi_0 = 200\,\mathrm{V}$. Figure 3 displays the distributions of transverse shear stresses, electric potential and electric displacement in the thickness direction for different slenderness ratios employing again seven SaS for each layer.

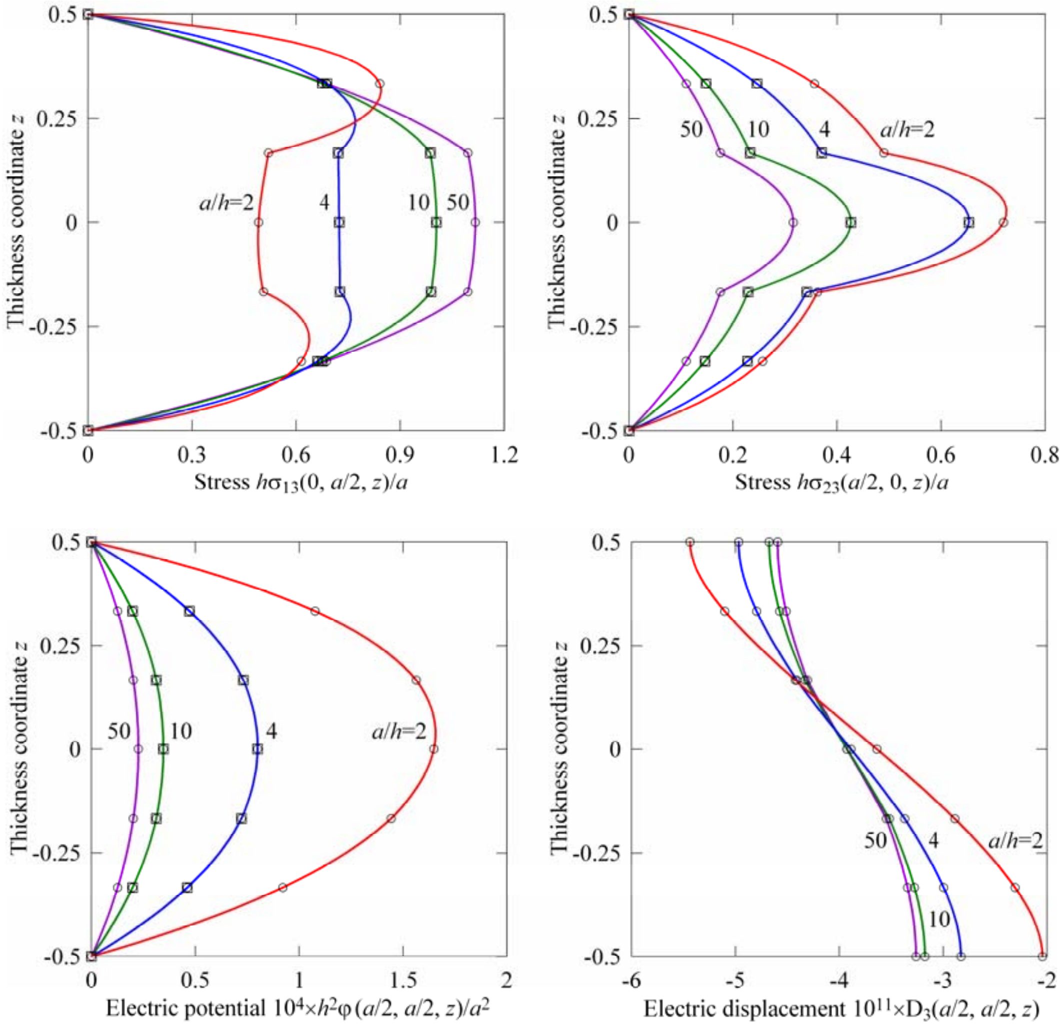

Figure 2: Through-thickness distributions of transverse shear stresses, electric potential and electric displacement for a three-layer plate subjected to mechanical loading for $I_1 = I_2 = I_3 = 7$: strong SaS formulation (—), variational SaS formulation [2] (○) and Heyliger's solution [3] (□), where $z = x_3 / h$

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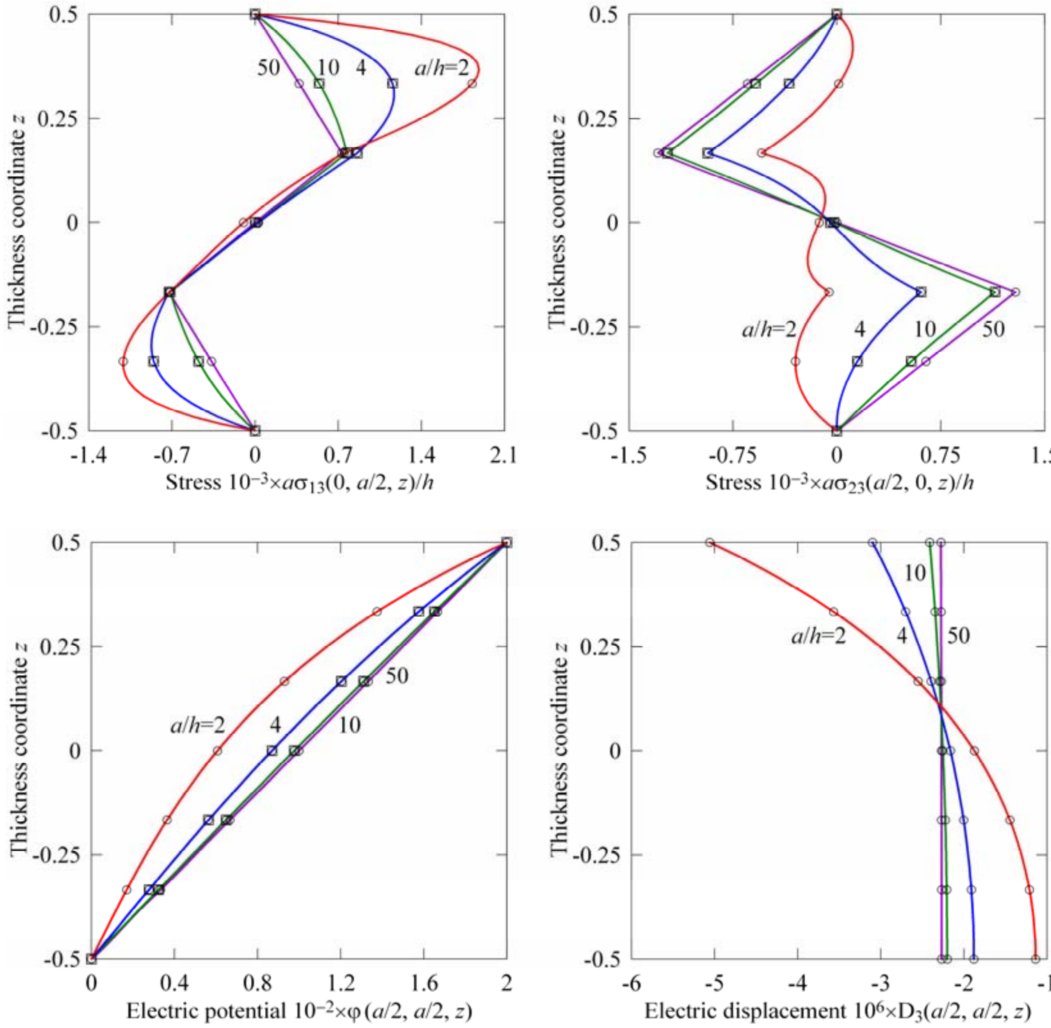

Figure 3: Through-thickness distributions of transverse shear stresses, electric potential and electric displacement for a three-layer plate subjected to electric loading for $I_1 = I_2 = I_3 = 7$: strong SaS formulation (—), variational SaS formulation [2] (○) and Heyliger's solution [3] (□), where $z = x_3 / h$

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