FINITE ELEMENT ANALYSIS OF LAMINATED PIEZOELECTRIC SHELLS VIA SAMPLING SURFACES FORMULATION

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Abstract. A hybrid-mixed ANS four-node piezoelectric shell element by using the sampling surfaces (SaS) technique is developed. The SaS formulation is based on choosing inside the nth layer I_n SaS located at Chebyshev polynomial nodes in order to introduce the displacements and electric potentials of these surfaces as basic shell unknowns. The interfaces and outer surfaces are also included into a set of SaS. Such choice of unknowns with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction for displacements, strains, electric potential and electric field allows the presentation of the laminated piezoelectric shell formulation in a very compact form. To implement the efficient analytical integration throughout the element, the enhanced ANS method is employed. The proposed hybrid-mixed four-node piezoelectric shell element is based on the Hu-Washizu variational equation and exhibits a superior performance in the case of coarse meshes. It could be useful for the 3D stress analysis of thick and thin doubly-curved layered piezoelectric shells since the SaS formulation gives the possibility to obtain numerical solutions with a prescribed accuracy, which asymptotically approach the exact solutions of elasticity as the number of SaS tends to infinity.

1 HU-WASHIZU VARIATIONAL SAS FORMULATION

Consider a laminated shell of the thickness h. Let the middle surface Ω be described by orthogonal curvilinear coordinates θ_1 and θ_2, which are referred to the lines of principal curvatures of its surface. The coordinate θ_3 is oriented along the unit vector normal to the middle surface. According to the SaS concept, we choose inside the nth layer I_n SaS $\Omega^{(n)1}, \Omega^{(n)2}, ..., \Omega^{(n)I_n}$ located at Chebyshev polynomial nodes and interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$

(see Figure 1), where the index $n = 1,2,...,N$ identifies the belonging of any quantity to the nth layer; N is the number of layers.

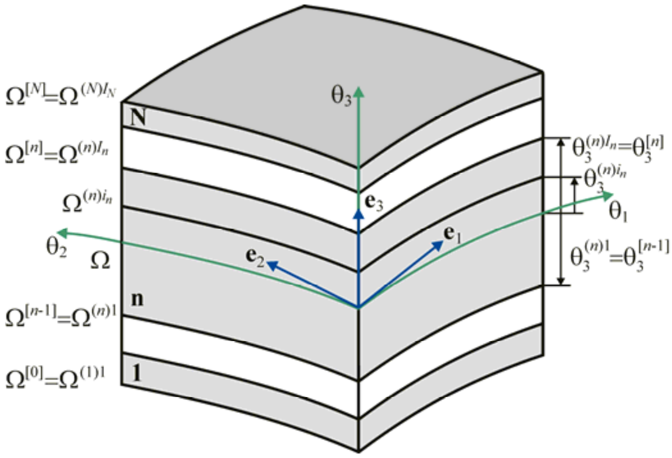

Figure 1: Geometry of the layered piezoelectric shell

The through- thickness SaS approximations [1] can be written as

$$[u_i^{(n)}\ \varepsilon_{ij}^{(n)}\ \sigma_{ij}^{(n)}\ \varphi^{(n)}\ E_i^{(n)}] = \sum_{i_n} L^{(n)i_n} [u_i^{(n)i_n}\ \varepsilon_{ij}^{(n)i_n}\ \sigma_{ij}^{(n)i_n}\ \varphi^{(n)i_n}\ E_i^{(n)i_n}], \tag{1}$$

where $u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}$ are the displacements, strains, stresses, electric potential and electric field of the nth layer; $u_i^{(n)i_n}, \varepsilon_{ij}^{(n)i_n}, \sigma_{ij}^{(n)i_n}, \varphi^{(n)i_n}, E_i^{(n)i_n}$ are the displacements, strains, stresses, electric potential and electric field of SaS of the nth layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(\theta_3)$ are the Lagrange basis polynomials of degree $I_n - 1$ related to the nth layer:

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{\theta_3 - \theta_3^{(n)j_n}}{\theta_3^{(n)i_n} - \theta_3^{(n)j_n}} \quad (i_n, j_n = 1,2,...,I_n), \tag{2}$$

where the indices i_n, j_n identify the belonging of any quantity to the SaS of the nth layer.

The proposed hybrid-mixed piezoelectric solid-shell element is based on the Hu-Washizu variational equation of electroelasticity in which displacements, strains and stresses are utilized as independent variables [2]:

$$\delta \iint_{\Omega} \sum_n \int_{\theta_3^{[n-1]}}^{\theta_3^{[n]}} \left[\frac{1}{2}\eta_{ij}^{(n)} C_{ijkl}^{(n)} \eta_{kl}^{(n)} - E_k^{(n)} e_{kij}^{(n)} \eta_{ij}^{(n)} - \frac{1}{2} E_i^{(n)} \in_{ij}^{(n)} E_j^{(n)} - \sigma_{ij}^{(n)} (\eta_{ij}^{(n)} - \varepsilon_{ij}^{(n)}) \right] dV = \delta W, \tag{3}$$

where $dV = A_1 A_2 (1 + k_1\theta_3)(1 + k_2\theta_3) d\theta_1 d\theta_2 d\theta_3$ is the infinitesimal volume element; A_1, A_2 and k_1, k_2 are the coefficients of the first fundamental form and principal curvatures of the middle surface; $\theta_3^{[n-1]}$ and $\theta_3^{[n]}$ are the transverse coordinates of interfaces; $\varepsilon_{ij}^{(n)}$ and $\eta_{ij}^{(n)}$ are the displacement-dependent and displacement-independent strains; $C_{ijkl}^{(n)}$, $e_{kij}^{(n)}$ and $\in_{ij}^{(n)}$ are the elastic, piezoelectric and dielectric constants; W is the work done by external electromechanical loads. Here, the summation on repeated Latin indices is implied.

Following the SaS technique (1), we introduce the next assumption of the hybrid-mixed solid-shell element formulation. Assume that the displacement-independent strains are distributed through the thickness of the nth layer as follows:

$$\eta_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} \eta_{ij}^{(n)i_n}, \tag{4}$$

where $\eta_{ij}^{(n)i_n}$ are the displacement-independent strains of SaS of the nth layer.

2 FINITE ELEMENT FORMULATION

The finite element formulation is based on a simple interpolation of the shell via exact geometry four-node piezoelectric solid-shell elements

$$u_i^{(n)i_n} = \sum_r N_r u_{ir}^{(n)i_n}, \quad \varphi^{(n)i_n} = \sum_r N_r \varphi_r^{(n)i_n}, \tag{5}$$

where $N_r(\xi_1, \xi_2)$ are the bilinear shape functions of the element; $u_{ir}^{(n)i_n}$ and $\varphi_r^{(n)i_n}$ are the displacements and electric potentials of SaS $\Omega^{(n)i_n}$ at element nodes; ξ_1, ξ_2 are the normalized curvilinear coordinates θ_1, θ_2; the nodal index r runs from 1 to 4. The term "exact geometry" reflects the fact that the parametrization of the middle surface is known a priori and, therefore, the coefficients of the first and second fundamental forms are taken exactly at element nodes.

To implement the efficient analytical integration throughout the element, the enhanced ANS method is employed:

$$\varepsilon_{ij}^{(n)i_n} = \sum_r N_r \varepsilon_{ijr}^{(n)i_n}, \quad E_i^{(n)i_n} = \sum_r N_r E_{ir}^{(n)i_n}, \tag{6}$$

where $\varepsilon_{ijr}^{(n)i_n}$ and $E_{ir}^{(n)i_n}$ are the displacement-dependent strains and electric field of SaS of the nth layer at element nodes. The main idea of such approach can be traced back to the ANS method developed by many scientists for the isoparametric finite element formulation. In contrast with above formulation, we treat the term "ANS" in a broader sense. In the proposed exact geometry four-node solid-shell element formulation, all components of the displacement-dependent strain tensor and electric field vector are assumed to vary bilinearly throughout the biunit square in (ξ_1, ξ_2)-space.

To overcome shear and membrane locking and have no spurious zero energy modes, the robust strain and stress interpolations [3] are utilized:

$$\left[\eta_{11}^{(n)i_n}\ \eta_{22}^{(n)i_n}\ \eta_{33}^{(n)i_n}\ \eta_{12}^{(n)i_n}\ \eta_{13}^{(n)i_n}\ \eta_{23}^{(n)i_n}\right]^{\mathrm{T}} = \mathbf{P}\boldsymbol{\Psi}^{(n)i_n}, \quad \boldsymbol{\Psi}^{(n)i_n} = \left[\Psi_1^{(n)i_n}\ \Psi_2^{(n)i_n}\ \ldots\ \Psi_{12}^{(n)i_n}\right]^{\mathrm{T}}, \tag{7}$$

$$\left[\sigma_{11}^{(n)i_n}\ \sigma_{22}^{(n)i_n}\ \sigma_{33}^{(n)i_n}\ \sigma_{12}^{(n)i_n}\ \sigma_{13}^{(n)i_n}\ \sigma_{23}^{(n)i_n}\right]^{\mathrm{T}} = \mathbf{P}\boldsymbol{\Xi}^{(n)i_n}, \quad \boldsymbol{\Xi}^{(n)i_n} = \left[\Xi_1^{(n)i_n}\ \Xi_2^{(n)i_n}\ \ldots\ \Xi_{12}^{(n)i_n}\right]^{\mathrm{T}}, \tag{8}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \xi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \xi_1 \end{bmatrix}$$

that provides a correct rank of the element stiffness matrix.

Substituting first the through-thickness SaS approximations (1), (4) and then the finite element interpolations (5)-(8) into the Hu-Washizu variational equation (3), we arrive at the element equilibrium equations. Eliminating the column matrices $\mathbf{\Psi}^{(n)i_n}$ and $\mathbf{\Xi}^{(n)i_n}$ on the element level, the following system of linear equations are obtained:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\varphi} \\ \mathbf{K}_{\varphi u} & \mathbf{K}_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_p \\ \mathbf{F}_q \end{bmatrix}, \tag{9}$$

where $\mathbf{K}_{uu}, \mathbf{K}_{u\varphi}, \mathbf{K}_{\varphi u} = (\mathbf{K}_{u\varphi})^{\mathrm{T}}, \mathbf{K}_{\varphi\varphi}$ are the element stiffness matrices; $\mathbf{U}$ and $\mathbf{\Phi}$ are the element displacement and electric potential vectors of order $12N_{\mathrm{SaS}}$ and $4N_{\mathrm{SaS}}$; $\mathbf{F}_p$ and $\mathbf{F}_q$ are the element-wise mechanical and electric surface vectors; $N_{\mathrm{SaS}} = \sum_n I_n - N + 1$ is the total number of SaS.

It is worth noting that the element stiffness matrices are evaluated without the expensive numerical matrix inversion that is impossible in available isoparametric hybrid-mixed finite element formulations.

3 NUMERICAL EXAMPLE

Consider a simply supported symmetric three-layer cylindrical shell with equal ply thicknesses under the imposed transverse deformation on the top surface

$$u_3^+ = u_0 \sin\frac{\pi\theta_1}{L}\cos 2\theta_2, \tag{10}$$

where L is the length of the shell and $u_0 = 10^{-8}\,\mathrm{m}$. The both outer layers are composed of the PZT-4 with the material properties given in [1, 4]. The middle layer is made of the fictitious material with the elastic constants exactly half of the PZT-4 and the piezoelectric and dielectric constants exactly double those of the PZT-4. The bottom and top surfaces are assumed to be electrically grounded. To compare the results with the exact solution of Heyliger [4], we take $L = R^+ = 0.01\,\mathrm{m}$, where R^+ is the radius of the top cylindrical surface.

Due to symmetry of the problem, only one octant of the shell ($L/2 \le \theta_1 \le L$, $0 \le \theta_2 \le \pi/2$) is modeled by a fine uniform mesh 48×96. To analyze the results, we introduce the following scaled variables:

$$\bar{u}_1 = 10^{11}\times u_1(L,\,0,\,z), \quad \bar{u}_3 = 10^{11}\times u_3(L/2,\,0,\,z),$$

$$\bar{\sigma}_{13} = 10^{-3} \times \sigma_{13}(L, 0, z), \quad \bar{\sigma}_{23} = 10^{-3} \times \sigma_{23}(L/2, \pi/4, z),$$
$$\bar{\varphi} = \varphi(L/2, 0, z), \quad \bar{D}_3 = 10^6 \times D_3(L/2, 0, z), \quad z = \theta_3 / h. \tag{11}$$

Figure 2 displays the distributions of displacements, transverse shear stresses, electric potential and electric displacement in the thickness direction for different values of the slenderness ratio $S = R^+ / h$ taking five SaS for each layer. A comparison with the exact SaS solution [1] and Heyliger's exact 3D solution [4] is also presented. These results demonstrate convincingly the high potential of the proposed hybrid-mixed SaS solid-shell element formulation. This is due to the fact that the boundary conditions on the bottom and top surfaces for transverse shear stresses and the continuity conditions at interfaces for transverse shear stresses and electric displacement are satisfied with a high accuracy.

Consider next the same symmetric three-layer cylindrical shell subjected to electric loading on the top surface whereas the bottom surface is electrically grounded:

$$\varphi^- = 0, \quad \varphi^+ = \varphi_0 \sin\frac{\pi\theta_1}{L}\cos 2\theta_2, \tag{12}$$

where $\varphi_0 = 10\,\mathrm{V}$. The bottom and top surfaces of the plate are assumed to be traction free. The obtained results are compared with the exact SaS solution [1]. Figure 3 shows the through-thickness distributions of displacements, transverse shear stresses, electric potential and electric displacement (11) in the thickness direction for different slenderness ratios S by choosing five SaS for each layer. It is seen that the boundary conditions on the bottom and top surfaces and continuity conditions at interfaces for transverse shear stresses and electric displacement are satisfied again correctly.

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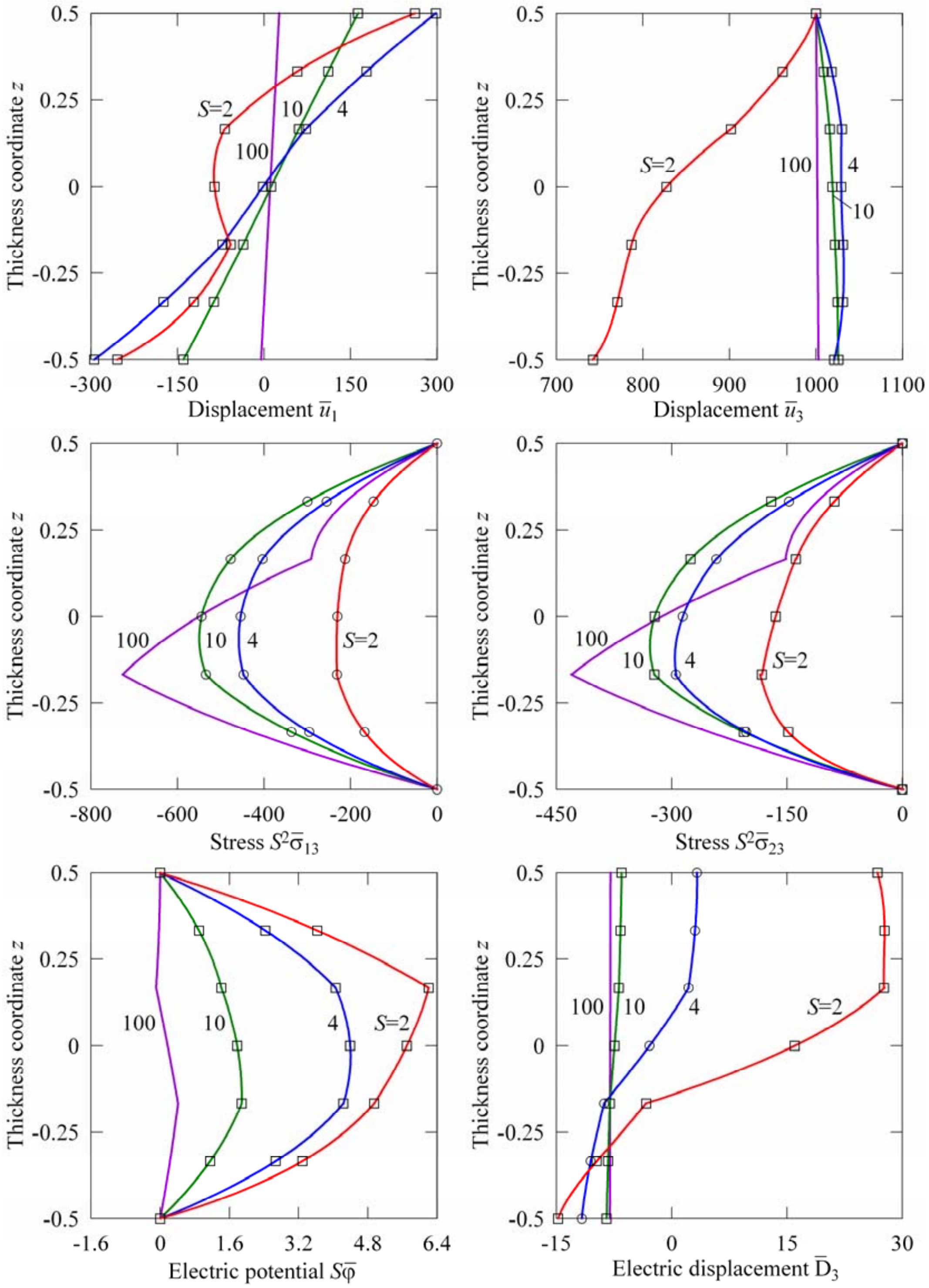

Figure 2: Through-thickness distributions of displacements, transverse shear stresses, electric potential and electric displacement for a three-layer cylindrical shell subjected to mechanical loading for $I_1 = I_2 = I_3 = 5$: exact geometry SaS solid-shell element (—), exact SaS solution [1] (○) and Heyliger's exact 3D solution [4] (□)

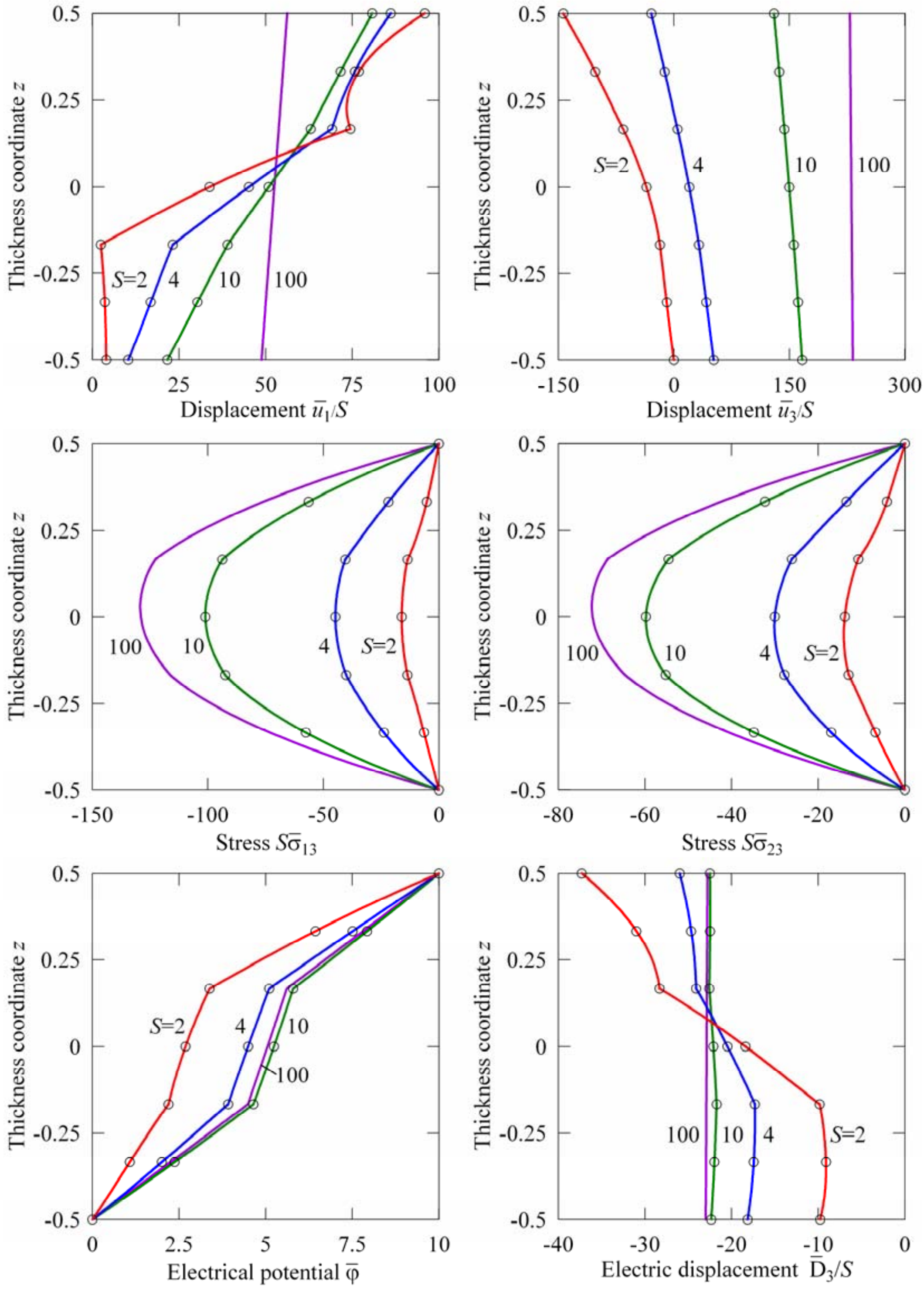

Figure 3: Through-thickness distributions of displacements, transverse shear stresses, electric potential and electric displacement for a three-layer cylindrical shell subjected to electric loading for $I_1 = I_2 = I_3 = 5$: exact geometry SaS solid-shell element (—) and exact SaS solution [1] (○)