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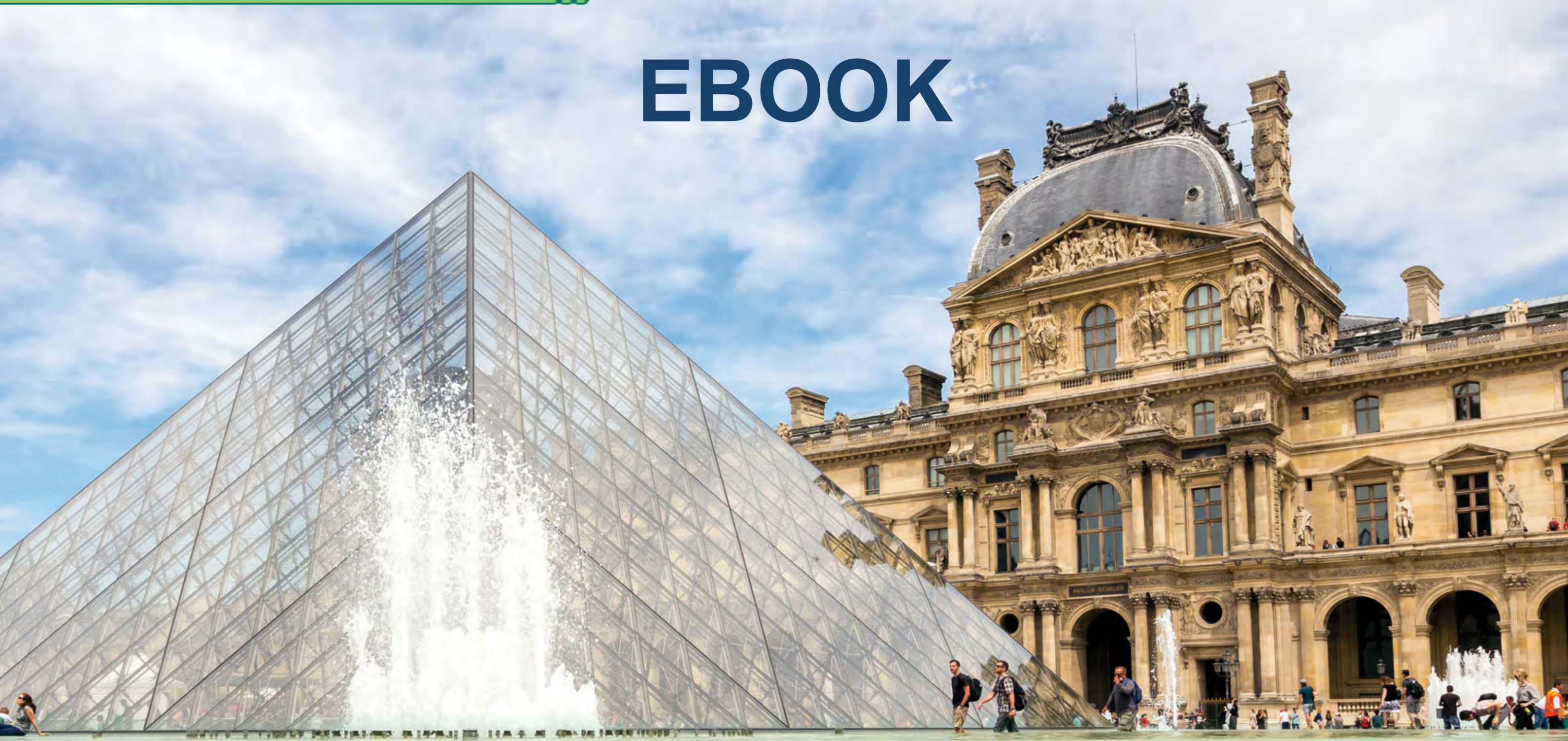

Paris, France, 8-11 July 2019

SMART 2019

9th ECCOMAS Thematic Conference on Smart Structures and Materials

Ayech Benjeddou, Nazih Mechbal and Jean-François Deü (Editors)

9th ECCOMAS Thematic Conference on

Smart Structures and Materials

SMART 2019

Paris, France
July 8 - 11, 2019

A publication of:

International Centre for Numerical Methods in Engineering (CIMNE)
Barcelona, Spain

Printed by: Artes Gráficas Torres S.L., Huelva 9, 08940 Cornellà de Llobregat, Spain

EXACT VIBRATION ANALYSIS OF LAMINATED PIEZOELECTRIC PLATES THROUGH STRONG SAS FORMULATION

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Key words: Piezoelectricity, 3D Vibration Analysis, Laminated Piezoelectric Plate, Sampling Surfaces Method.

Abstract. This paper focuses on implementation of the sampling surfaces (SaS) method for the 3D vibration analysis of laminated piezoelectric plates. The SaS formulation is based on choosing inside the layers the arbitrary number of SaS parallel to the middle surface to introduce the displacements and electric potentials of these surfaces as basic plate variables. Such choice of unknowns allows the presentation of the laminated piezoelectric plate formulation in a very compact form. The feature of the proposed approach is that all SaS are located inside the layers at Chebyshev polynomial nodes that improves the convergence of the SaS method significantly. The use of outer surfaces and interfaces is avoided that makes possible to minimize uniformly the error due to Lagrange interpolation. Therefore, the strong SaS formulation based on direct integration of the equations of motion and the charge equation can be applied efficiently to the obtaining of exact solutions for laminated piezoelectric plates, which asymptotically approach the 3D solutions of piezoelectricity as the number of SaS tends to infinity.

1 INTRODUCTION

The exact vibration analysis of laminated piezoelectric plates was first carried out by Heyliger and Brooks [1], and Heyliger and Saravanos [2] using the Pagano approach. The most popular state space approach was utilized for the free vibration of simply supported electroelastic plates in works [3-7]. Messina and Carrera [8] proposed to employ the transfer matrix method to solve the ordinary differential equations in terms of the displacements and electric potential derived from the system of partial differential equations through the separating variable procedure. The dynamic response of laminated piezoelectric plates by a Taylor series expansion through the thickness was studied in papers [9-11]. The SaS approach was also used for the free vibration analysis of piezolaminated plates [12].

The SaS method [13] has been applied effectively to the 3D stress analysis of laminated piezoelectric structures by Kulikov and Plotnikova [14-17]. According to this method, we choose the arbitrary number of SaS throughout the layers parallel to the middle surface and located at Chebyshev polynomial nodes in order to introduce the displacements and electric potentials of these surfaces as basic plate unknowns. Such choice of unknowns with the consequent use of Lagrange polynomials in the through-thickness distributions of displacements, strains, electric potential and electric field leads to a robust laminated piezoelectric plate formulation. The above works are based on the variational SaS formulation, which requires including the interfaces into a set of SaS. However, it is important to take all SaS located at Chebyshev polynomial nodes due to the convergence criterion [18].

The present paper is intended to extend the variational SaS formulation for the free vibration of laminated piezoelectric plates [12] to the strong SaS formulation. The latter is based on the choice of all SaS inside the layers at Chebyshev polynomial nodes and direct integration of the equations of motion and the charge equation. The use of interfaces is avoided that allows one to minimize uniformly the error due to the higher-order Lagrange interpolation [19-22]. Thus, the strong SaS formulation can be applied efficiently to the 3D vibration analysis of piezolaminated plates.

2 BASIC ASSUMPTIONS

Consider a laminated piezoelectric plate of the thickness h. Let the middle surface Ω be described by Cartesian coordinates x_1 and x_2. The coordinate x_3 is oriented in the thickness direction. According to the SaS concept, we choose inside the nth layer I_n SaS $\Omega^{(n)1}, \Omega^{(n)2}, ..., \Omega^{(n)I_n}$ parallel to the middle surface (see Figure 1), where $n = 1, 2, ..., N$; N is the number of layers and $I_n \geq 3$. The transverse coordinates of SaS of the nth layer located at Chebyshev polynomial nodes (roots of the Chebyshev polynomial of order I_n) are written as

$$x_3^{(n)i_n} = \frac{1}{2}(x_3^{[n-1]} + x_3^{[n]}) - \frac{1}{2} h_n \cos\left(\pi \frac{2i_n - 1}{2I_n}\right), \tag{1}$$

where $x_3^{[0]} = -h/2$, $x_3^{[N]} = h/2$; $x_3^{[m]}$ are the transverse coordinates of interfaces $\Omega^{[m]}$; $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the nth layer; the index $m = 1, 2, ..., N-1$ identifies the belonging of any quantity to the interface; the indices $i_n, j_n = 1, 2, ..., I_n$ identify the belonging of any quantity to the SaS of the nth layer.

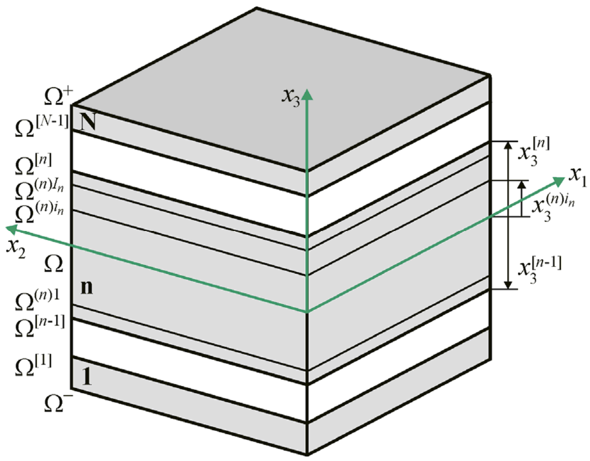

Figure 1: Geometry of the laminated piezoelectric plate

The through-thickness SaS approximations can be expressed as

$$[u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}, D_i^{(n)}] = \sum_{i_n} L^{(n)i_n} [u_i^{(n)i_n}, \varepsilon_{ij}^{(n)i_n}, \sigma_{ij}^{(n)i_n}, \varphi^{(n)i_n}, E_i^{(n)i_n}, D_i^{(n)i_n}], \quad (2)$$

where $u_i^{(n)}, \varepsilon_{ij}^{(n)}, \sigma_{ij}^{(n)}, \varphi^{(n)}, E_i^{(n)}, D_i^{(n)}$ are the displacements, strains, stresses, electric potential, electric field and electric displacements of the *n*th layer; $u_i^{(n)i_n}, \varepsilon_{ij}^{(n)i_n}, \sigma_{ij}^{(n)i_n}, \varphi^{(n)i_n}, E_i^{(n)i_n}, D_i^{(n)i_n}$ are the displacements, strains, stresses, electric potential, electric field and electric displacements of SaS of the *n*th layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(x_3)$ are the Lagrange basis polynomials of degree $I_n - 1$ corresponding to the *n*th layer:

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}}. \quad (3)$$

3 STRONG SAS FORMULATION

For simplicity, we consider the case of linear piezoelectric materials given by

$$\sigma_{ij}^{(n)} = C_{ijkl}^{(n)} \varepsilon_{kl}^{(n)} - e_{kij}^{(n)} E_k^{(n)}, \quad (4)$$

$$D_i^{(n)} = e_{ikl}^{(n)} \varepsilon_{kl}^{(n)} + \in_{ik}^{(n)} E_k^{(n)}, \quad (5)$$

where $C_{ijkl}^{(n)}$, $e_{kij}^{(n)}$ and $\in_{ik}^{(n)}$ are the elastic, piezoelectric and dielectric constants of the *n*th layer. Here, the summation on repeated Latin indices is implied.

The equations of motion and the charge equation of the laminated piezoelectric plate are written as

$$\sigma_{ij,j}^{(n)} = \rho_n \ddot{u}_i^{(n)}, \quad (6)$$

$$D_{i,i}^{(n)} = 0, \quad (7)$$

where ρ_n is the mass density of the *n*th layer; $\ddot{u}_i^{(n)}$ is the second order derivative of displacements with respect to time *t*; the symbol $(\ldots)_{,i}$ stands for the partial derivatives with respect to coordinates x_i.

The boundary conditions on bottom and top surfaces are defined as

$$u_i^{(1)}(-h/2) = w_i^- \text{ or } \sigma_{i3}^{(1)}(-h/2) = p_i^-, \quad \varphi^{(1)}(-h/2) = \phi^- \text{ or } D_3^{(1)}(-h/2) = Q^-, \quad (8)$$

$$u_i^{(N)}(h/2) = w_i^+ \text{ or } \sigma_{i3}^{(N)}(h/2) = p_i^+, \quad \varphi^{(N)}(h/2) = \phi^+ \text{ or } D_3^{(N)}(h/2) = Q^+, \quad (9)$$

where $w_i^-, p_i^-, \phi^-, Q^-$ and $w_i^+, p_i^+, \phi^+, Q^+$ are the prescribed displacements, surface tractions, electric potentials and electric charges at the bottom and top surfaces.

The continuity conditions at interfaces are

$$u_i^{(m)}(x_3^{[m]}) = u_i^{(m+1)}(x_3^{[m]}), \quad \sigma_{i3}^{(m)}(x_3^{[m]}) = \sigma_{i3}^{(m+1)}(x_3^{[m]}), \quad (10)$$

$$\varphi^{(m)}(x_3^{[m]}) = \varphi^{(m+1)}(x_3^{[m]}), \quad D_3^{(m)}(x_3^{[m]}) = D_3^{(m+1)}(x_3^{[m]}). \quad (11)$$

Satisfying equations of motion (6) and charge equation (7) at inner layer points $x_3^{(n)m_n}$ by using the SaS approximations (2), the following differential equations are obtained:

$$\sigma_{i1,1}^{(n)m_n} + \sigma_{i2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n})\sigma_{i3}^{(n)i_n} = \rho_n \ddot{u}_i^{(n)m_n}, \tag{12}$$

$$D_{1,1}^{(n)m_n} + D_{2,2}^{(n)m_n} + \sum_{i_n} M^{(n)i_n}(x_3^{(n)m_n}) D_3^{(n)i_n} = 0, \tag{13}$$

where $M^{(n)i_n} = L_{,3}^{(n)i_n}$ are the derivatives of the Lagrange basis polynomials whose values at SaS $\Omega^{(n)m_n}$ are presented in papers [14, 15]; $m_n = 2,3,...,I_n - 1$.

Next, we satisfy the boundary conditions on bottom and top surfaces

$$\sum_{i_1} L^{(1)i_1}(-h/2) u_i^{(1)i_1} = w_i^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2)\sigma_{i3}^{(1)i_1} = p_i^-, \tag{14}$$

$$\sum_{i_1} L^{(1)i_1}(-h/2)\varphi^{(1)i_1} = \phi^- \ \text{ or } \ \sum_{i_1} L^{(1)i_1}(-h/2) D_3^{(1)i_1} = Q^-,$$

$$\sum_{i_N} L^{(N)i_N}(h/2) u_i^{(N)i_N} = w_i^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2)\sigma_{i3}^{(N)i_N} = p_i^+, \tag{15}$$

$$\sum_{i_N} L^{(N)i_N}(h/2)\varphi^{(1)i_N} = \phi^+ \ \text{ or } \ \sum_{i_N} L^{(N)i_N}(h/2) D_3^{(N)i_N} = Q^+,$$

and the continuity conditions at interfaces

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) u_i^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) u_i^{(m+1)i_{m+1}}, \tag{16}$$

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) \sigma_{i3}^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) \sigma_{i3}^{(m+1)i_{m+1}},$$

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) \varphi^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) \varphi^{(m+1)i_{m+1}}, \tag{17}$$

$$\sum_{i_m} L^{(m)i_m}(x_3^{[m]}) D_3^{(m)i_m} = \sum_{i_{m+1}} L^{(m+1)i_{m+1}}(x_3^{[m]}) D_3^{(m+1)i_{m+1}}.$$

Thus, the proposed strong SaS formulation deals with $4N_{\text{SaS}}$ governing equations (12)-(17) for obtaining the same number of SaS displacements $u_i^{(n)i_n}$ and SaS electric potentials $\varphi^{(n)i_n}$, where $N_{\text{SaS}} = I_1 + I_2 + ... + I_N$ is the total number of SaS. These differential and algebraic equations have to be solved to describe the dynamic response of the laminated piezoelectric plate.

4 FREE VIBRATION OF SIMPLY SUPPORTED PIEZOELECTRIC PLATE

In this section, we consider a laminated piezoelectric rectangular plate with simply supported edges. The boundary conditions on the edges are written as

$$\sigma_{11}^{(n)} = u_2^{(n)} = u_3^{(n)} = \varphi^{(n)} = 0 \ \text{ at } \ x_1 = 0 \ \text{ and } \ x_1 = a, \tag{18}$$

$$\sigma_{22}^{(n)} = u_1^{(n)} = u_3^{(n)} = \varphi^{(n)} = 0 \ \text{ at } \ x_2 = 0 \ \text{ and } \ x_2 = b,$$

where *a* and *b* are the length and width of the plate.

To satisfy boundary conditions (18), we seek the analytical solution of the problem in the following form:

$$u_1^{(n)i_n} = u_{1rs}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \cos \bar{r}x_1 \sin \bar{s}x_2, \quad u_2^{(n)i_n} = u_{2rs}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \cos \bar{s}x_2,$$
$$u_3^{(n)i_n} = u_{3rs}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \sin \bar{s}x_2, \quad \varphi^{(n)i_n} = \varphi_{rs}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \sin \bar{s}x_2, \tag{19}$$

where $\bar{r} = r\pi / a, \quad \bar{s} = s\pi / b$; r and s are the half-wave numbers in x_1 and x_2 directions; $u_{irs}^{(n)i_n}$ and $\varphi_{rs}^{(n)i_n}$ are the amplitudes of displacements and electric potentials of SaS; ω_{rs} is the circular frequency; $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

Using (19) in relations between the SaS variables [12], one finds

$$(\varepsilon_{11}^{(n)i_n}, \varepsilon_{22}^{(n)i_n}, \varepsilon_{33}^{(n)i_n}, \sigma_{11}^{(n)i_n}, \sigma_{22}^{(n)i_n}, \sigma_{33}^{(n)i_n}) = (\varepsilon_{11rs}^{(n)i_n}, \varepsilon_{22rs}^{(n)i_n}, \varepsilon_{33rs}^{(n)i_n}, \sigma_{11rs}^{(n)i_n}, \sigma_{22rs}^{(n)i_n}, \sigma_{33rs}^{(n)i_n}) \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \sin \bar{s}x_2,$$
$$(E_3^{(n)i_n}, D_3^{(n)i_n}) = (E_{3rs}^{(n)i_n}, D_{3rs}^{(n)i_n}) \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \sin \bar{s}x_2,$$
$$(\varepsilon_{13}^{(n)i_n}, \sigma_{13}^{(n)i_n}, E_1^{(n)i_n}, D_1^{(n)i_n}) = (\varepsilon_{13rs}^{(n)i_n}, \sigma_{13rs}^{(n)i_n}, E_{1rs}^{(n)i_n}, D_{1rs}^{(n)i_n}) \mathrm{e}^{\mathrm{i}\omega_{rs}t} \cos \bar{r}x_1 \sin \bar{s}x_2, \tag{20}$$
$$(\varepsilon_{23}^{(n)i_n}, \sigma_{23}^{(n)i_n}, E_2^{(n)i_n}, D_2^{(n)i_n}) = (\varepsilon_{23rs}^{(n)i_n}, \sigma_{23rs}^{(n)i_n}, E_{2rs}^{(n)i_n}, D_{2rs}^{(n)i_n}) \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin \bar{r}x_1 \cos \bar{s}x_2,$$
$$(\varepsilon_{12}^{(n)i_n}, \sigma_{12}^{(n)i_n}) = (\varepsilon_{12rs}^{(n)i_n}, \sigma_{12rs}^{(n)i_n}) \mathrm{e}^{\mathrm{i}\omega_{rs}t} \cos \bar{r}x_1 \cos \bar{s}x_2,$$

where

$$\varepsilon_{11rs}^{(n)i_n} = -\bar{r}u_{1rs}^{(n)i_n}, \quad \varepsilon_{22rs}^{(n)i_n} = -\bar{s}u_{2rs}^{(n)i_n}, \quad 2\varepsilon_{12rs}^{(n)i_n} = \bar{s}u_{1rs}^{(n)i_n} + \bar{r}u_{2rs}^{(n)i_n},$$
$$2\varepsilon_{13rs}^{(n)i_n} = \bar{r}u_{3rs}^{(n)i_n} + \beta_{1rs}^{(n)i_n}, \quad 2\varepsilon_{23rs}^{(n)i_n} = \bar{s}u_{3rs}^{(n)i_n} + \beta_{2rs}^{(n)i_n}, \quad \varepsilon_{33rs}^{(n)i_n} = \beta_{3rs}^{(n)i_n},$$
$$\beta_{irs}^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n}) u_{irs}^{(n)j_n}, \tag{21}$$
$$E_{1rs}^{(n)i_n} = -\bar{r}\varphi_{rs}^{(n)i_n}, \quad E_{2rs}^{(n)i_n} = -\bar{s}\varphi_{rs}^{(n)i_n}, \quad E_{3rs}^{(n)i_n} = -\sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n}) \varphi_{rs}^{(n)j_n}.$$

In the case of the piezoelectric material with *4mm* symmetry, the constitutive equations (4) and (5) can be written in terms of SaS variables

$$\sigma_{ijrs}^{(n)i_n} = C_{ijkl}^{(n)} \varepsilon_{klrs}^{(n)i_n} - e_{kij}^{(n)} E_{krs}^{(n)i_n}, \tag{22}$$

$$D_{irs}^{(n)i_n} = e_{ikl}^{(n)} \varepsilon_{klrs}^{(n)i_n} + \in_{ik}^{(n)} E_{krs}^{(n)i_n}. \tag{23}$$

For the vibration analysis of piezoelectric plates with stress-free and voltage-free external surfaces, the boundary conditions (14) and (15) are used with $p_i^{\pm} = 0$ and $\phi^{\pm} = 0$. Substituting (19)-(23) in governing equations (12)-(17), we arrive at the homogeneous system of linear equations

$$\left(\begin{bmatrix} \mathbf{K}_{rs}^{uu} & \mathbf{K}_{rs}^{u\varphi} \\ \mathbf{K}_{rs}^{\varphi u} & \mathbf{K}_{rs}^{\varphi\varphi} \end{bmatrix} - \omega_{rs}^2 \begin{bmatrix} \mathbf{M}_{rs} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{U}_{rs} \\ \mathbf{\Phi}_{rs} \end{bmatrix} = \mathbf{0}, \tag{24}$$

where $\mathbf{K}_{rs}^{uu}$, $\mathbf{K}_{rs}^{u\varphi}$, $\mathbf{K}_{rs}^{\varphi u} = (\mathbf{K}_{rs}^{u\varphi})^{\mathrm{T}}$ and $\mathbf{K}_{rs}^{\varphi\varphi}$ are the mechanical, piezoelectric and dielectric stiffness matrices; $\mathbf{M}_{rs}$ is the mass matrix; $\mathbf{U}_{rs}$ is the SaS displacement vector of order $3N_{\mathrm{SaS}}$; $\mathbf{\Phi}_{rs}$ is the SaS electric potential vector of order N_{SaS} given by

$$\mathbf{U}_{rs} = \left[\mathbf{U}_{1rs}^{\mathrm{T}}\ \mathbf{U}_{2rs}^{\mathrm{T}}\ \mathbf{U}_{3rs}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{25}$$

$$\mathbf{U}_{irs} = \left[u_{irs}^{(1)1}\ u_{irs}^{(1)2} \ldots u_{irs}^{(1)I_1}\ u_{irs}^{(2)1}\ u_{irs}^{(2)2} \ldots u_{irs}^{(2)I_2} \ldots u_{irs}^{(N)1}\ u_{irs}^{(N)2} \ldots u_{irs}^{(N)I_N}\right]^{\mathrm{T}},$$

$$\mathbf{\Phi}_{rs} = \left[\varphi_{rs}^{(1)1}\ \varphi_{rs}^{(1)2} \ldots \varphi_{rs}^{(1)I_1}\ \varphi_{rs}^{(2)1}\ \varphi_{rs}^{(2)2} \ldots \varphi_{rs}^{(2)I_2} \ldots \varphi_{rs}^{(N)1}\ \varphi_{rs}^{(N)2} \ldots \varphi_{rs}^{(N)I_N}\right]^{\mathrm{T}}. \tag{26}$$

Eliminating the vector $\mathbf{\Phi}_{rs}$ from (24), one gets

$$\mathbf{\Phi}_{rs} = -(\mathbf{K}_{rs}^{\varphi\varphi})^{-1}\mathbf{K}_{rs}^{\varphi u}\mathbf{U}_{rs}. \tag{27}$$

Inserting (27) in the first row of (24), the following reduced homogeneous system is obtained

$$(\mathbf{K}_{rs} - \omega_{rs}^2\mathbf{M}_{rs})\mathbf{U}_{rs} = \mathbf{0}, \tag{28}$$

which has a non-trivial solution only if

$$\det(\mathbf{K}_{rs} - \omega_{rs}^2\mathbf{M}_{rs}) = 0, \tag{29}$$

where $\mathbf{K}_{rs} = \mathbf{K}_{rs}^{uu} - \mathbf{K}_{rs}^{u\varphi}(\mathbf{K}_{rs}^{\varphi\varphi})^{-1}\mathbf{K}_{rs}^{\varphi u}$ is the stiffness matrix of order $3N_{\mathrm{SaS}} \times 3N_{\mathrm{SaS}}$.

The polynomial equation (29) has to be solved to obtain the circular frequencies $0 < \omega_{rs}^{(1)} < \omega_{rs}^{(2)} < \ldots < \omega_{rs}^{(3N_{\mathrm{SaS}}-6N)}$ arranged in an increasing order. The number of frequencies $\omega_{rs}^{(q)}$ for each set of SaS depends on the number of zero rows in a mass matrix $\mathbf{M}$, where the superscript $q = 1,2,\ldots,3N_{\mathrm{SaS}} - 6N$ stands for the number of through thickness modes. The eigenvectors $\mathbf{U}_{rs}^{(q)}$ associated with the corresponding eigenvalues $\lambda_{rs}^{(q)} = (\omega_{rs}^{(q)})^2$ can be evaluated by using the linear system (28).

5 FORCED VIBRATION OF SIMPLY SUPPORTED PIEZOELECTRIC PLATE

Here, we study forced vibrations of the simply supported laminated piezoelectric rectangular plate with boundary conditions on the bottom and top surfaces

$$\sigma_{13}^{(1)} = \sigma_{23}^{(1)} = \sigma_{33}^{(1)} = \varphi^{(1)} = 0 \ \text{ at } \ x_3 = -h/2, \tag{30}$$
$$\sigma_{13}^{(N)} = \sigma_{23}^{(N)} = 0, \quad \sigma_{33}^{(N)} = p_3^+, \quad \varphi^{(N)} = \phi^+ \ \text{ at } \ x_3 = h/2.$$

Consider time-harmonic loading distributed on the top surface as follows:

$$\text{Problem A:} \quad p_3^+ = p_0 \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \quad \phi^+ = 0; \tag{31}$$

$$\text{Problem B:} \quad p_3^+ = 0, \quad \phi^+ = \phi_0 \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \tag{32}$$

where ω is the forcing frequency.

To satisfy boundary conditions (18), we seek the analytical solution of the problem in the following form:

$$u_1^{(n)i_n} = u_{10}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega t} \cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \quad u_2^{(n)i_n} = u_{20}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{b}, \tag{33}$$

$$u_3^{(n)i_n} = u_{30}^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \quad \varphi^{(n)i_n} = \varphi_0^{(n)i_n} \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b},$$

where $u_{i0}^{(n)i_n}$ and $\varphi_0^{(n)i_n}$ are the amplitudes of displacements and electric potentials of SaS of the nth layer.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. This makes possible to obtain the analytical solutions for free and forced vibrations of the simply supported laminated piezoelectric rectangular plate in the framework of the SaS formulation, which asymptotically approaches the 3D exact solutions of electroelasticity as the number of SaS goes to infinity.

As a numerical example, we consider a simply supported two-ply square plate [0/90] made of the graphite epoxy composite and covered with PZT-4 piezoelectric layers at the bottom and at the top. Therefore, we deal here with a hybrid four-layer plate [PZT/0/90/PZT] with ply thicknesses [0.25h/0.25h/0.25h/0.25h]. The material properties of the PZT-4 [12] polarized in the thickness direction are E_1=E_2=81.3 GPa, E_3=64.5 GPa, G_{12}=30.6 GPa, G_{13}=G_{23}=25.6 GPa, ν_{12}=0.329, ν_{13}=ν_{23}=0.432, e_{311}=e_{322}=-5.2 C/m^2, e_{333}=15.08 C/m^2, e_{113}=e_{223}=12.72 C/m^2, $\epsilon_{11}=\epsilon_{22}=$ 13.06 nF/m, $\epsilon_{33}=$ 11.51 nF/m and ρ=7600 kg/m^3. The material properties of the graphite epoxy [12] are E_1=172.5 GPa, E_2=E_3=6.9 GPa, G_{12}=G_{13}=3.45 GPa, G_{23}=1.38 GPa, ν_{12}=ν_{13}=0.25, ν_{23}=0.35, $\epsilon_{11}=$ 0.031 nF/m, $\epsilon_{22}=\epsilon_{33}=$ 0.027 nF/m and ρ=1800 kg/m^3.

To evaluate the results effectively, we introduce the dimensionless frequency [12]

$$\bar{\omega} = \omega a^2 \sqrt{\rho_0 / E_0} / h \tag{34}$$

and dimensionless basic variables at crucial points as functions of the thickness coordinate

$$\begin{gathered} \bar{u}_3 = 10^9 u_3(a/2,a/2,z)/h, \quad \bar{\sigma}_{11} = \sigma_{11}(a/2,a/2,z)/p_0, \\ \bar{\sigma}_{13} = \sigma_{13}(0,a/2,z)/p_0, \quad \bar{\sigma}_{33} = \sigma_{33}(a/2,a/2,z)/p_0, \\ \bar{\varphi} = \varphi(a/2,a/2,z)/\phi_0, \quad \bar{D}_3 = 10^9 \phi_0 D_3(a/2,a/2,z)/ap_0, \quad z = x_3/h, \end{gathered} \tag{35}$$

where a=1 m, h=0.1 m, E_0=81.3 GPa, ρ_0=7600 kg/m^3, p_0=1 Pa and $\phi_0 = 1$ V.

Figures 2 and 3 display the distributions of displacements, stresses, electric potential and electric displacement (35) through the thickness of the plate for the forcing frequencies $\bar{\omega} = 0,\ 0.8\bar{\omega}_0,\ 0.95\bar{\omega}_0$ and $1.05\bar{\omega}_0$ using seven SaS inside each layer, where $\bar{\omega}_0 = 6.0932$ is the fundamental frequency in the case of stress-free and voltage-free external surfaces [12]. It is seen that the boundary conditions on bottom and top surfaces for the transverse stresses and the continuity conditions at interfaces for the transverse stresses and electric displacement are satisfied correctly. Note also that the displacements and stresses become larger as the forcing frequency approaches the fundamental frequency.

ACKNOWLEDGEMENTS

This work was supported by the Russian Science Foundation under Grant No. 18-19-00092.

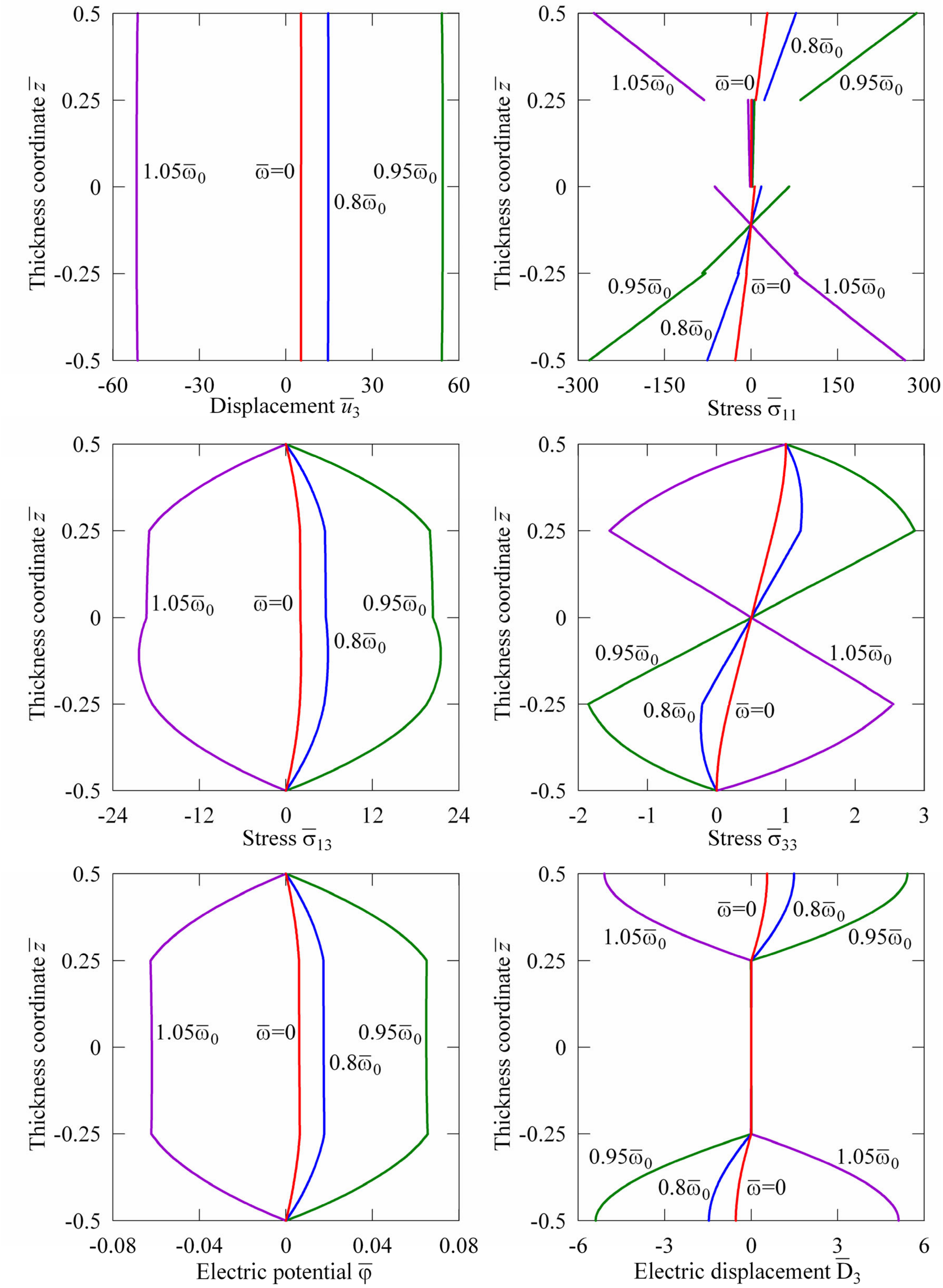

Figure 2: Through-thickness distributions of transverse displacement, stresses, electric potential and electric displacement for the four-layer plate subjected to mechanical loading (problem A)

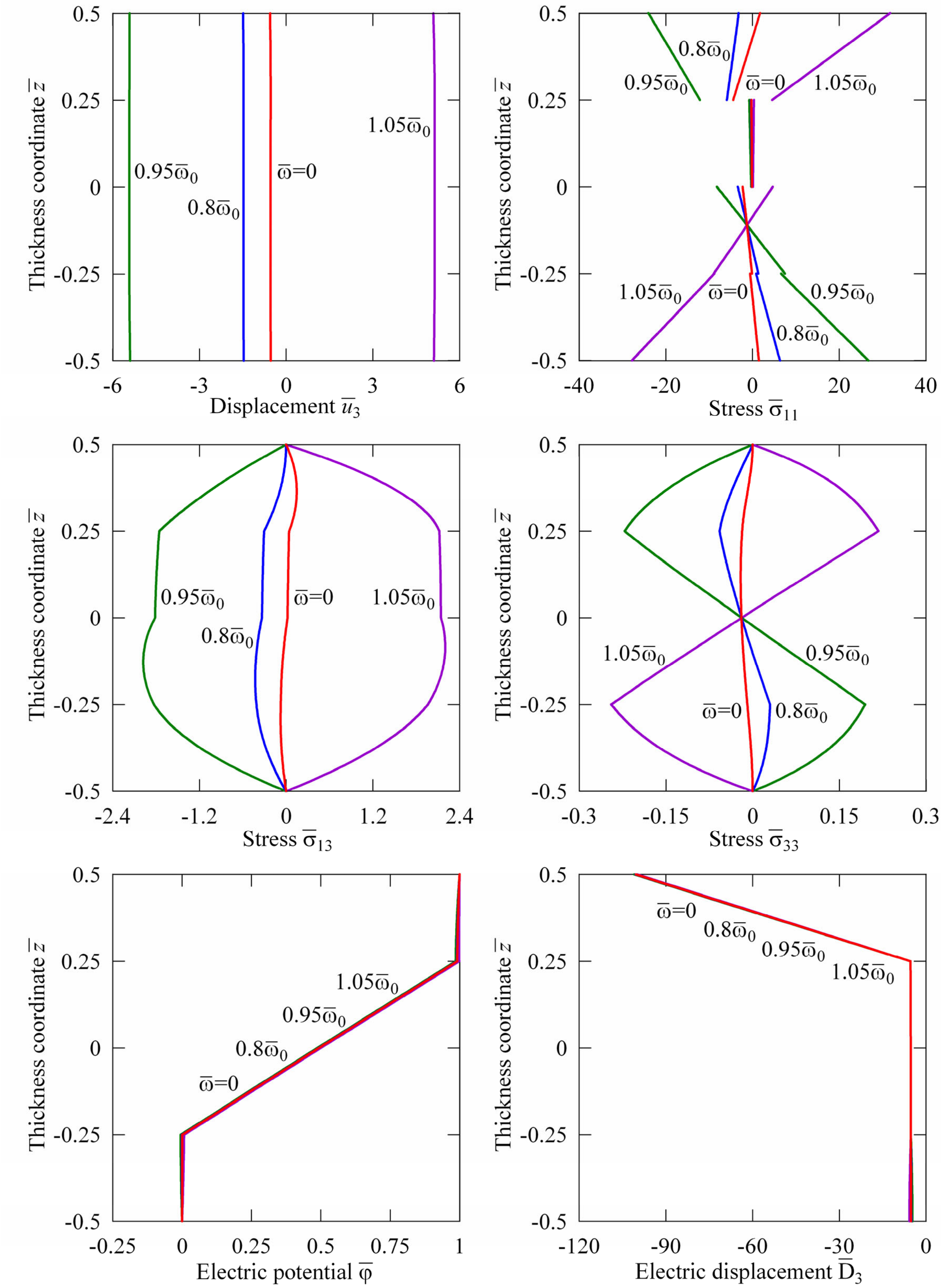

Figure 3: Through-thickness distributions of transverse displacement, stresses, electric potential and electric displacement for the four-layer plate subjected to electric loading (problem B)

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