Shell Structures: Theory and Applications, Vol. 2 – Pietraszkiewicz & Kreja (eds)
© 2010 Taylor & Francis Group, London, ISBN 978-0-415-54883-0

Non-linear geometrically exact solid-shell element under follower loads

G.M. Kulikov & S.V. Plotnikova
Tambov State Technical University, Tambov, Russia

ABSTRACT: This paper focuses on the formulation of non-linear geometrically exact four-node solid-shell elements based on the 7-parameter shell model, which permits to utilize the 3D constitutive equations. As fundamental shell unknowns six displacements of the outer surfaces and a transverse displacement of the midsurface are chosen. To avoid shear and membrane locking and have no spurious zero energy modes, the assumed strain and stress resultant fields are invoked. To improve a geometrically non-linear shell response, the modified ANS method is applied. Additionally, analytical integration throughout the element is employed to evaluate the tangent stiffness matrix. As a result, the present finite rotation solid-shell element formulation allows one to analyze thin-walled shell structures subjected to dead and follower loads by using coarse meshes and very large load increments.

1 INTRODUCTION

A large number of works has been already done to develop the finite rotation *isoparametric* solid-shell element formulation based on the 7-parameter shell model accounting for thickness stretching (Parisch 1995, Brank 2005). In the isoparametric solid-shell element formulation, initial and deformed geometry are equally interpolated allowing one to describe rigid-body shell motions precisely. Still, the isoparametric solid-shell element formulation is computationally inefficient because stresses and strains are analyzed in the global and local orthogonal Cartesian coordinate systems, although the normalized element coordinates represent already convected curvilinear coordinates.

An alternative way is to develop non-linear *geometrically exact* solid-shell elements based on presentation of displacement vectors in the reference surface frame (Arciniega & Reddy 2007, Kulikov 2007, Kulikov & Plotnikova 2008) that finds its point of departure in the paper of Kulikov & Plotnikova (2003) devoted to the 6-parameter shell formulation. The term "geometrically exact" reflects the fact that coefficients of the first and second fundamental forms are taken exactly at every integration point. Therefore, no approximation of the reference surface is needed. The feature of the Kulikov & Plotnikova's geometrically exact solid-shell element formulation is that it is based on the strain-displacement relationships, which precisely represent arbitrarily large rigid-body shell motions in a convected curvilinear coordinate system.

Herein, a more general 7-parameter geometrically exact solid-shell element formulation is considered, which permits to analyze thin-walled shell structures subjected to dead and follower loads efficiently. As fundamental unknowns six displacements of the outer surfaces and a transverse displacement of the middle surface are chosen. Taking into account that displacement vectors of outer and middle surfaces of the shell are resolved in the reference surface frame, the proposed geometrically exact solid-shell element formulation has computational advantages compared to the conventional isoparametric solid-shell element formulations, since it reduces the computational cost of numerical integration in evaluation of the tangent stiffness matrix. Additionally, we use the efficient 3D analytical integration (Kulikov & Plotnikova 2008) that gives the possibility to employ coarse meshes.

2 KINEMATIC DESCRIPTION OF SHELL

Let us consider a shell of the thickness h. The shell can be defined as a 3D body bounded by two outer surfaces Ω^- and Ω^+, located at the distances d^- and d^+ measured with respect to the reference surface Ω such that $h = d^- + d^+$, and the edge boundary surface S. Let the reference surface be referred to the orthogonal curvilinear coordinates θ_1 and θ_2, which referred to the lines of principal curvatures of its surface, whereas the coordinate θ_3 is oriented along the unit vector $\boldsymbol{a}_3 = \mathbf{e}_3$ normal to the reference surface; $\boldsymbol{a}_\alpha = A_\alpha \mathbf{e}_\alpha$ are the base vectors of the reference surface; $\mathbf{g}_\alpha^I = A_\alpha^I \mathbf{e}_\alpha$ are the base vectors of outer and middle surfaces; $\mathbf{e}_\alpha$ are the unit vectors tangent to the lines of principal curvatures; A_α are the Lamé coefficients of the reference surface; $A_\alpha^I = A_\alpha c_\alpha^I$ are the Lamé coefficients of outer and middle surfaces; $c_\alpha^I = 1 + k_\alpha z^I$ are the components of the shifter tensor at outer and middle surfaces; k_α are the principal curvatures of the reference surface; $z^- = -d^-$, $z^+ = d^+$ and z^{M} are the transverse coordinates of outer and middle surfaces. Here and in the following developments, Greek indices α, β range from 1 to 2; indices i, j range from 1 to 3; Greek indices

A, B identify the belonging of any quantity to the bottom and top surfaces and take values $-$ and $+$; indices I, J identify the belonging of any quantity to the outer and middle surfaces and take values $-$, $+$ and M.

The base vectors of outer surfaces in the current shell configuration are given by

$$\bar{\mathbf{g}}_\alpha^A = \mathbf{g}_\alpha^A + \mathbf{u}_{,\alpha}^A \tag{1}$$

$$\mathbf{u}^A = \sum_i u_i^A \mathbf{e}_i \tag{2}$$

where $\mathbf{u}^A(\theta_1, \theta_2)$ = displacement vectors of outer surfaces.

Taking into account Equation 1, we can find the unit vectors $\mathbf{n}^A$ normal to the outer surfaces in the current shell configuration:

$$\mathbf{n}^A = \frac{1}{\left|\bar{\mathbf{g}}_1^A \times \bar{\mathbf{g}}_2^A\right|} \bar{\mathbf{g}}_1^A \times \bar{\mathbf{g}}_2^A \tag{3}$$

The use of Equations 1 and 2 and well-known formulas for the derivatives of orthonormal base vectors $\mathbf{e}_i$ with respect to curvilinear coordinates θ_α into Equation 3 leads to a more convenient formula:

$$\mathbf{n}^A = \frac{1}{\eta^A} \sum_i \eta_i^A \mathbf{e}_i \tag{4}$$

$$\eta_1^A = \lambda_{21}^A \lambda_{32}^A - \lambda_{31}^A (c_2^A + \lambda_{22}^A), \quad \eta_2^A = \lambda_{12}^A \lambda_{31}^A - \lambda_{32}^A (c_1^A + \lambda_{11}^A)$$

$$\eta_3^A = (c_1^A + \lambda_{11}^A)(c_2^A + \lambda_{22}^A) - \lambda_{12}^A \lambda_{21}^A$$

$$\eta^A = \sqrt{(\eta_1^A)^2 + (\eta_2^A)^2 + (\eta_3^A)^2}$$

where

$$\lambda_{\alpha\alpha}^A = \left(\frac{1}{A_\alpha} u_\alpha^A\right)_{,\alpha} + B_{\alpha\alpha} u_\alpha^A + B_{\alpha\beta} u_\beta^A + k_\alpha u_3^A \tag{5}$$

$$\lambda_{\beta\alpha}^A = \left(\frac{1}{A_\alpha} u_\beta^A\right)_{,\alpha} + B_{\alpha\alpha} u_\beta^A - B_{\alpha\beta} u_\alpha^A \quad (\beta \neq \alpha)$$

$$\lambda_{3\alpha}^A = \left(\frac{1}{A_\alpha} u_3^A\right)_{,\alpha} + B_{\alpha\alpha} u_3^A - k_\alpha u_\alpha^A, \quad B_{\alpha\beta} = \frac{1}{A_\alpha A_\beta} A_{\alpha,\beta}$$

Since the follower pressure load is assumed to be normal to the outer surfaces in the current shell configuration, it changes as a shell undergoes deformation. Thus, Equations 4 and 5 are of great importance to the shell formulation developed because with the help of these equations one can recalculate external loads *at each step* of the Newton-Raphson iteration process. This permits to reduce significantly the number of loading increments.

The displacement field is approximated in the thickness direction according to Kulikov (2001):

$$u_\alpha = \sum_A N^A u_\alpha^A, \quad u_3 = \sum_I L^I u_3^I \tag{6}$$

$$N^- = \frac{1}{h}(z^+ - \theta_3), \quad N^+ = \frac{1}{h}(\theta_3 - z^-)$$

where $u_i^A(\theta_1, \theta_2)$ = components of the displacement vectors of outer surfaces; $u_3^M(\theta_1, \theta_2)$ = transverse displacement of the middle surface Ω^M; $L^I(\theta_3)$ = Lagrange polynomials of the second orders such that $L^I(z^J) = 1$ for $J = I$ and $L^I(z^J) = 0$ for $J \neq I$.

The non-linear strain-displacement relationships of the first-order 7-parameter shell model (Kulikov & Plotnikova 2008) can be written as

$$\varepsilon_{\alpha\beta} = \sum_A N^A \varepsilon_{\alpha\beta}^A, \quad \varepsilon_{33} = \sum_A N^A \varepsilon_{33}^A \tag{7}$$

$$\varepsilon_{\alpha 3} = \hat{\varepsilon}_{\alpha 3}, \quad \hat{\varepsilon}_{\alpha 3} = \frac{1}{2}(\varepsilon_{\alpha 3}^- + \varepsilon_{\alpha 3}^+)$$

where $\varepsilon_{ij}^A(\theta_1, \theta_2)$ = components of the Green-Lagrange strain tensor of outer surfaces defined in (Kulikov & Plotnikova 2008). It is noteworthy that the Green-Lagrange strain components (7) are objective, i.e., they represent precisely all large rigid-body shell motions in any convected curvilinear coordinate system (Kulikov & Plotnikova 2008).

3 FINITE ELEMENT FORMULATION

The hybrid/mixed geometrically exact solid-shell element formulation developed is based on the assumed approximations of displacements (6) and displacement-dependent strains (7) in the thickness direction. Additionally, to circumvent shear and membrane locking, we introduce the similar approximation for the assumed displacement-independent strains:

$$\varepsilon_{\alpha\beta}^{AS} = \sum_A N^A E_{\alpha\beta}^A, \quad \varepsilon_{33}^{AS} = \sum_A N^A E_{33}^A, \quad \varepsilon_{\alpha 3}^{AS} = E_{\alpha 3} \tag{8}$$

where $E_{ij}^A(\theta_1, \theta_2)$ = displacement-independent strains of outer surfaces, since a convenient notation $E_{\alpha 3}^A = E_{\alpha 3}$ is utilized.

For the four-node *curved* solid-shell element the displacement field is approximated according to the standard C^0 interpolation:

$$\mathbf{v} = \sum_r N_r \mathbf{v}_r, \quad \mathbf{v} = [u_1^- \; u_2^- \; u_3^- \; u_1^+ \; u_2^+ \; u_3^+ \; u_3^M]^T \tag{9}$$

$$\mathbf{v}_r = [u_{1r}^- \; u_{2r}^- \; u_{3r}^- \; u_{1r}^+ \; u_{2r}^+ \; u_{3r}^+ \; u_{3r}^M]^T$$

where $\mathbf{v}_r$ = displacement vectors of the element nodes; $N_r(\xi_1, \xi_2)$ = bilinear shape functions of the element; ξ_α = normalized curvilinear coordinates θ_α; the index r denotes a number of nodes and ranges from 1 to 4.

To avoid shear and membrane locking and have no spurious zero energy modes, the assumed displacement-independent strains are approximated inside the element as follows:

$$E_{ij}^A = \sum_{r_1, r_2} \alpha_{ij}^{r_1 r_2} (\xi_1)^{r_1} (\xi_2)^{r_2} E_{ij}^{A r_1 r_2} \tag{10}$$

The non-vanishing components $\alpha_{ij}^{r_1r_2}$ are given by $\alpha_{ij}^{00} = 1$ and $\alpha_{11}^{01} = \alpha_{13}^{01} = \alpha_{33}^{01} = \alpha_{22}^{10} = \alpha_{23}^{10} = \alpha_{33}^{10} = \alpha_{33}^{11} = 1$. Here and below, the indices r_1, r_2 run from 0 to 1.

For the stress resultants (Kulikov & Plotnikova 2008)

$$H_{\alpha\beta}^{A} = \int_{z^-}^{z^+} S_{\alpha\beta} N^A d\theta_3,\; H_{33}^{A} = \int_{z^-}^{z^+} S_{33} N^A d\theta_3,\; H_{\alpha 3}^{A} = \int_{z^-}^{z^+} S_{\alpha 3} d\theta_3$$

we accept a similar approximation:

$$H_{ij}^{A} = \sum_{r_1,r_2} \alpha_{ij}^{r_1r_2} (\xi_1)^{r_1} (\xi_2)^{r_2} H_{ij}^{Ar_1r_2} \tag{11}$$

where S_{ij} = components of the second Piola-Kirchhoff stress tensor.

The use of Equation 9 in the strain-displacement relationships (Kulikov & Plotnikova 2008) yields the biquadratic interpolation for displacement-dependent strains:

$$\boldsymbol{\varepsilon} = \sum_{s_1,s_2} (\xi_1)^{s_1} (\xi_2)^{s_2} (\mathbf{B}^{s_1s_2} + \mathbf{A}^{s_1s_2}\mathbf{V})\mathbf{V} \tag{12}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{11}^{-}\; \varepsilon_{11}^{+}\; \varepsilon_{22}^{-}\; \varepsilon_{22}^{+}\; \varepsilon_{33}^{-}\; \varepsilon_{33}^{+}\; 2\varepsilon_{12}^{-}\; 2\varepsilon_{12}^{+}\; 2\hat{\varepsilon}_{13}\; 2\hat{\varepsilon}_{23}]^{T}$$

where $\mathbf{V} = [\mathbf{v}_1^T \mathbf{v}_2^T \mathbf{v}_3^T \mathbf{v}_4^T]^T$ = displacement vector of the shell element; $\mathbf{B}^{s_1s_2}$ = constant matrices of order 10 × 28 corresponding to the linear strain-displacement transformation such that $\mathbf{B}^{s_1s_2} = \mathbf{0}$ for $s_1 = 2$ or $s_2 = 2$; $\mathbf{A}^{s_1s_2}$ = constant 3D arrays of order 10 × 28 × 28 corresponding to the non-linear strain-displacement transformation such that products $\mathbf{A}^{s_1s_2}\mathbf{V}$ generate the matrices of order 10 × 28 (Kulikov & Plotnikova 2008).

However, a described non-linear geometrically exact four-node solid-shell element is too stiff in the case of coarse mesh configurations. To improve a geometrically non-linear response of the shell, we invoke the ANS method using its non-conventional form (Kulikov & Plotnikova 2008):

$$\boldsymbol{\varepsilon}^{ANS} = \sum_{r} N_r \boldsymbol{\varepsilon}(P_r) \tag{13}$$

where P_r = element nodes.

For simplicity we limit our discussion to the case of uniform follower pressure p^- applied to the bottom surface of the shell but the general case also has been included in the finite element formulation. Due to this assumption, the virtual work done by the follower pressure load is expressed as

$$\delta W_{ext}^{-} = p^{-} \iint_{\Omega^-} \sum_{i} \delta u_i^{-} n_i^{-} dS \tag{14}$$

where n_i^- = direction cosines of the outward pointing normal to the deformed bottom surface. Substituting Equation 9 in Equation 14 and taking into account Equations 4 and 5, one can represent the virtual work as follows:

$$\delta W_{ext}^{-} = \delta \mathbf{V}^T p^- \mathbf{G}^-(\mathbf{V}) \tag{15}$$

where $\mathbf{G}^-(\mathbf{V})$ = column matrix depending only on the nodal displacements of the bottom surface u_{ir}^-.

Employing the incremental formulation:

$$^{t+\Delta t}\mathbf{V} = {}^{t}\mathbf{V} + \Delta\mathbf{V},\quad {}^{t+\Delta t}\mathbf{F} = {}^{t}\mathbf{F} + \Delta\mathbf{F},\quad {}^{t+\Delta t}p^- = {}^{t}p^- + \Delta p^- \tag{16}$$

in conjunction with the Newton-Raphson iteration scheme:

$$\Delta\mathbf{V}^{[n+1]} = \Delta\mathbf{V}^{[n]} + \Delta\mathcal{V}^{[n]} \quad (n = 0, 1, \ldots) \tag{17}$$

and following Kulikov & Plotnikova (2008), we obtain the linearized element equilibrium equations:

$$\mathbf{K}\Delta\mathcal{V}^{[n]} = \Delta\mathcal{F}^{[n]} \tag{18}$$

$$+({}^{t}p^- + \Delta p^-)\,\mathbf{G}^-({}^{t}\mathbf{V} + \Delta\mathbf{V}^{[n]}) - {}^{t}p^-\mathbf{G}^-({}^{t}\mathbf{V})$$

where $\Delta\mathbf{F}$ = incremental element-wise surface force vector due to dead loading; $\Delta\mathcal{F}^{[n]}$ = column matrix introduced by Kulikov & Plotnikova (2008); $\mathbf{K}$ = tangent stiffness matrix of order 24 × 24 defined as

$$\mathbf{K} = \mathbf{K}_D + \mathbf{K}_H + \mathbf{K}_L \tag{19}$$

The matrices $\mathbf{K}_D$ and $\mathbf{K}_H$ of the proposed hybrid/mixed finite element formulation are symmetric (Kulikov & Plotnikova 2008), whereas the load correction matrix $\mathbf{K}_L$ given by

$$\mathbf{K}_L = -({}^{t}p^- + \Delta p^-)\frac{\partial \mathbf{G}^-}{\partial \mathbf{V}}({}^{t}\mathbf{V} + \Delta\mathbf{V}^{[n]}) \tag{20}$$

is in general unsymmetric; see discussion on this subject in (Cohen 1966, Argyris & Symeonidis 1981, Schweizerhof & Ramm 1984).

The equilibrium equations (18) for each element are assembled by the usual technique to form the global incremental equilibrium equations. These incremental equations have to be performed until the required accuracy of the solution will be achieved. The convergence criterion used herein can be described as follows:

$$\left\|\Delta\mathbf{U}^{[n+1]} - \Delta\mathbf{U}^{[n]}\right\| < \varepsilon \left\|\Delta\mathbf{U}^{[n]}\right\| \tag{21}$$

where $\|\ldots\|$ = Euclidean norm in the displacement space; $\Delta\mathbf{U}$ = global vector of displacement increments; ε = prescribed tolerance.

The performance of proposed geometrically exact four-node solid-shell elements with the follower load action (GEX7P4F) and without (GEX7P4) are evaluated by means of the relatively simple problem. Consider a circular ring under non-uniform follower pressure $p(\varphi) = p_0(1 - e\cos 2\varphi)$. The geometrical and material characteristics of the ring are displayed in Figure 1. Due to symmetry of the problem, only

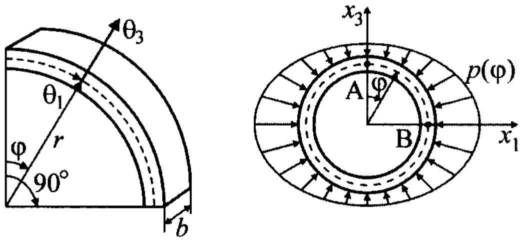

Figure 1. Circular ring under follower loading with $r = 100, h = 1, b = 20, E = 2.1 \times 10^7, \nu = 0.3$.

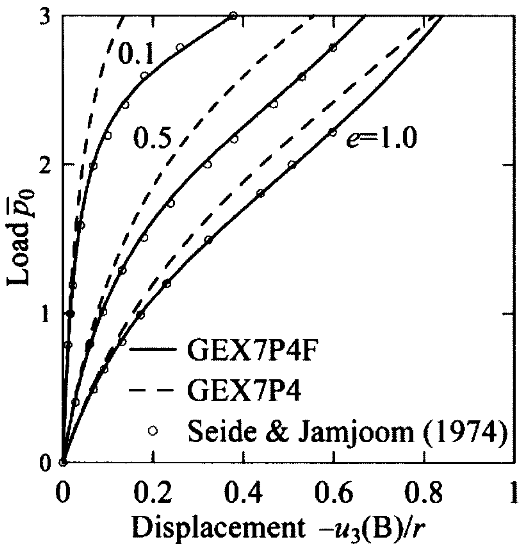

Figure 2. Midline displacement at point B of a ring.

one quarter of the ring is modeled by 18 × 1 meshes of GEX7P4 and GEX7P4F elements. A comparison with the analytical solution (Seide & Jamjoom 1974), presented by Argyris & Symeonidis (1981), is given in Figure 2, where $\bar{p}_0 = 12p_0r^3/Ebh^3$. As can be seen, the results obtained by using the GEX7P4F element are in good agreement with analytical ones. At the same time the GEX7P4 element is too stiff especially for small values of the eccentricity parameter *e*. Table 1 lists the transverse midline displacement at points A and B calculated for different values of NStep. Here, NStep denotes the number of load steps employed to *equally* divide the maximum load, whereas NIter stands for the total number of Newton iterations. It is seen that both proposed geometrically exact solid-shell elements are insensitive to the number of loading steps and a GEX7P4F element requires more Newton iterations.

Table 1. Displacement $\bar{u}_3 = u_3/r$ at points A and B of a ring under follower loading for $\bar{p}_0 = 3$ and $e = 1$ with $\varepsilon = 10^{-6}$.

	NStep = 1			NStep = 10		
Element	$\bar{u}_3$(A)	$-\bar{u}_3$(B)	NIter	$\bar{u}_3$(A)	$-\bar{u}_3$(B)	NIter
GEX7P4F	0.3660	0.8407	7	0.3660	0.8407	50
GEX7P4	0.3554	0.8220	6	0.3554	0.8220	30

ACKNOWLEDGEMENTS

The support of this work by RFBR under Grant No 08-01-00373 and Russian Ministry of Education and Science under Grant No 2.1.1/660 is gratefully acknowledged.

REFERENCES

Arciniega, R.A. & Reddy, J.N. 2007. Tensor-based finite element formulation for geometrically nonlinear analysis of shell structures. *Computer Methods in Applied Mechanics and Engineering* **196**: 1048–1073.

Argyris, J.H. & Symeonidis, S. 1981. Nonlinear finite element analysis of elastic systems under nonconservative loading – natural formulation. Part I. Quasistatic problems. *Computer Methods in Applied Mechanics and Engineering* **26**: 75–123.

Brank, B. 2005. Nonlinear shell models with seven kinematic parameters. *Computer Methods in Applied Mechanics and Engineering* **194**: 2336–2362.

Cohen, G.A. 1966. Conservativeness of a normal pressure field acting on a shell. *AIAA Journal* **4**:1886.

Kulikov, G.M. 2001. Refined global approximation theory of multilayered plates and shells. *Journal of Engineering Mechanics* **127**: 119–125.

Kulikov, G.M. 2007. On the first-order seven-parameter plate theory. *Trans. Tambov State Technical University* **13**: 518–528

Kulikov, G.M. & Plotnikova, S.V. 2003. Non-linear strain-displacement equations exactly representing large rigid-body motions. Part I. Timoshenko-Mindlin shell theory. *Computer Methods in Applied Mechanics and Engineering* **192**: 851–875.

Kulikov, G.M. & Plotnikova, S.V. 2008. Finite rotation geometrically exact four-node solid-shell element with seven displacement degrees of freedom. *Computer Modeling in Engineering & Sciences* **28**: 15–38.

Parisch, H. 1995. A continuum-based shell theory for non-linear applications. *International Journal for Numerical Methods in Engineering* **38**: 1855–1883.

Schweizerhof, K. & Ramm, E. 1984. Displacement dependent pressure loads in nonlinear finite element analyses. *Computers & Structures* **18**: 1099–1114.

Seide, P. & Jamjoom, T.M.M. 1974. Large deformations of circular rings under nonuniform normal pressure. *Journal of Applied Mechanics* **41**: 192–196.