G. M. Kulikov[1]

Computational Models for Multilayered Composite Shells with Application to Tires

REFERENCE: Kulikov, G. M., "**Computational Models for Multilayered Composite Shells with Application to Tires,**" *Tire Science and Technology,* TSTCA, Vol. 24, No. 1, January–March, 1996, pp. 11–38.

ABSTRACT: This paper focuses on four tire computational models based on two-dimensional shear deformation theories, namely, the first-order Timoshenko-type theory, the higher-order Timoshenko-type theory, the first-order discrete-layer theory, and the higher-order discrete-layer theory. The joint influence of anisotropy, geometrical nonlinearity, and laminated material response on the tire stress-strain fields is examined. The comparative analysis of stresses and strains of the cord-rubber tire on the basis of these four shell computational models is given. Results show that neglecting the effect of anisotropy leads to an incorrect description of the stress-strain fields even in bias-ply tires.

KEY WORDS: multilayered composite shell, computational models, tire, stress, strain

Pneumatic tires are the most widely used composite structures of commercial importance today. Pneumatic tires demand the careful investigation of their strength at the designing stage, which requires the development of mathematical models, computational algorithms, and computer programs for tires under different types of loading. It is necessary to emphasize that satisfactorily solving the strength problems of pneumatic tires is only possible on the basis of a theory taking into account the spatial character of the stress-strain fields, the effects of anisotropy, and the geometrical nonlinearity in tires modeled by multilayered shells of revolution with complicated shapes ([*1*,*2*]). The reference surface of such shells is formed by the revolution of an arbitrary curve given on a plane by a discrete number of points that have coordinates with random errors of measurement.

Early tire models based on elementary structural analyses are discussed in [*3*–*6*]; so, it is not necessary to consider them here in detail.

Although the network theory was developed as early as 1913 in the fundamental monograph by Haas and Dietzius [*7*] in connection with the progress in airship building, nevertheless, the application of this theory to the study

[1] Tambov State Technical University, 106 Sovetskaya Street, Tambov 392620, Russia.

of the simplest tire problem, concerning the inflated profile of a pneumatic tire, was obtained relatively recently [*8,9*]. Considering the principal proposition of the network theory, the governing equations of membrane shells of revolution subjected to an inflation pressure q can be presented in the following form [*8*]:

$$\frac{T_{11}^*}{R_1^*} + \frac{T_{22}^*}{R_2^*} = q \qquad T_{11}^* = \frac{(y^*)^2 - (y_0^*)^2}{2y^* \cos \alpha^*} q \qquad \frac{T_{22}^*}{T_{11}^*} = tg^2 \gamma_c^* \tag{1}$$

where T_{11}^* and T_{22}^* are the stress resultants in the meridional and circumferential directions; R_1^* and R_2^* are the radii of curvature in the meridional and circumferential directions; y^* is the distance from the rotation axis to a given point; y_0^* is the distance from the rotation axis to the widest point of the profile where the normal is parallel to the rotation axis; α^* is the angle between the tangent and the rotation axis; γ_c^* is the angle between the cord and meridian direction. The values with superscript $*$ have to do with the deformed profile of the tire. Putting $y_0^* = 0$ into Eq 1, we arrive at the original equations of the Haas-Dietzius' network theory [*7*]. The limitation is obviously connected with the shape of the airship because the widest point of its profile falls on the axis of rotation.

In spite of some shortcomings, such as neglecting the bending of the tire and the stiffening effects of rubber, and the limitations on the mechanical and geometrical characteristics of the layers (the number that must be even, arranged identically, and produced from the same materials), cord-network models allow the determination of the inflated profile of some bias-ply tires with the needed accuracy. When it became clear that the cord-network model applied to the study of the aircraft tires often led to unacceptable results, the researchers' attention was concentrated on the development of qualitatively new mathematical models for pneumatic tires [*4,5,10,11*]. Soon, membrane models for bias-ply tires, based on the momentless Kirchhoff-Love theory of orthotropic shells were created [*12–15*]. These models allow us to take into consideration not only the rubber influence but also the differences in mechanical and geometrical characteristics of cord-rubber plies.

The transition from bias-ply tires to radial tires required the development of computational models for radial tires. Apparently, for the first time, models were built [*14,16*] where a radial tire was modeled by a membrane shell composed of two layers simulating the belt and body plies in the tread area and a single layer in the sidewall (Fig. 1). The shortcomings of these membrane models are obvious because they cannot handle discontinuities in loading and material properties.

Considerable progress was made by Brewer [*17*]; he modeled tires by

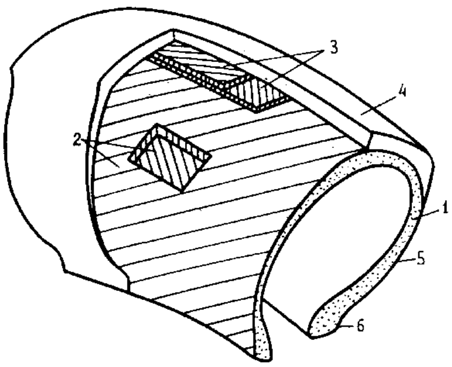

FIG. 1 — *Bias-Belted tire: body (1), body plies (2), belt plies (3), tread (4), sidewall (5), bead (6).*

applying the geometrically nonlinear Kirchhoff-Love theory of multilayered orthotropic shells.

The first results on pneumatic tire analysis, taking into account the transverse shear strains, have been obtained in [*18*]. In this paper, the sidewall of a radial tire is modeled by a homogeneous, transversely isotropic membrane shell and the tread area is modeled by a sandwich shell composed of two orthotropic membrane layers simulating the belt and body plies and rubber core between them (Fig. 1). Such tire modeling permitted the approximate calculation of interlaminar shear strains in the area between the belt plies and the tire body.

Qualitatively new tire computational models based on the geometrically nonlinear first-order Timoshenko-type theory of multilayered anisotropic (nonorthotropic) shells were developed in [*19–21*]. A more general computational tire model can be constructed by using the geometrically nonlinear higher-order Timoshenko-type theory of multilayered anisotropic shells as in [22]. This tire model, unlike the tire model in [*19*], allows to describe the nonlinear dependence of tangential stresses and strains on the thickness coordinate. This is especially true for the calculation of the stress-strain fields in the area of the belt edge.

More accurate tire computational models were built by the application of the discrete-layer theories of multilayered anisotropic shells [*23–26*]. In these models, the order of the governing differential equations is dependent on the number of tire layers. This allows the investigation of the complicated character of the distribution of transverse stresses in the tire thickness direction.

Two-Dimensional Models for Multilayered Anisotropic Shells

We will now consider all two-dimensional models for multilayered shells based on the so-called method of hypotheses and usually used in tire structural mechanics. For a detailed description of the models see [2,27,28]. The models may be grouped into the following five groups.

Kirchhoff-Love Model

Earlier literature on multilayered anisotropic shells was based on the classical Kirchhoff-Love theory, which has the following underlying assumptions:

$$\epsilon_{i3}^{(k)} = 0 \qquad \epsilon_{33}^{(k)} = 0 \tag{2}$$

where $\epsilon_{i3}^{(k)}$ and $\epsilon_{33}^{(k)}$ are the transverse shear and normal strains of the kth layer[2]; N is the total number of layers of the shell. Mathematically, this implies that the set of equations of the elasticity theory is supplemented by $3N$ new equations. The set is now overdetermined; therefore, we will have to drop $3N$ equations. The $3N$ relations of the generalized Hooke's law for transverse strains are discarded. The Kirchhoff-Love theory supposes that Hooke's relation for tangential strains $\epsilon_{ij}^{(k)}$ do not include the terms containing the transverse normal stresses $\sigma_{33}^{(k)}$, which are negligible compared with the tangential stresses $\sigma_{ij}^{(k)}$, that is,

$$\sigma_{33}^{(k)} << \sigma_{ij}^{(k)} \tag{3}$$

This static hypothesis is not of principal character because it introduces no new equations into the original set of equations of the elasticity theory.

Integrating Cauchy's relationships between the strains $\epsilon_{i3}^{(k)}$ and $\epsilon_{33}^{(k)}$ and displacements $u_i^{(k)}$ and $u_3^{(k)}$ of an arbitrary point of the kth layer, using Eq 2 and the continuity conditions for displacements at the layer interfaces, we arrive at the kinematic Kirchhoff hypothesis

$$u_i^{(k)} = u_i + z\theta_i \qquad u_3^{(k)} = w \qquad \theta_i = R_i^{-1}u_i - A_i^{-1}\partial w/\partial\alpha_i \tag{4}$$

where u_i and w are the tangential and transverse displacements of the reference surface of the shell; α_i is the orthogonal curvilinear coordinate, which coincides with the lines of the principal curvatures of the reference surface; z is the coordinate normal to the reference surface; R_i and A_i are the principal radii of curvature and the Lamé coefficients of the reference surface, respectively. The kinematic Kirchhoff hypothesis implies that the straight line elements, normal to the reference surface of the shell before deformation, remain straight and normal during deformation and retain their original length.

In this model, the order of the governing system of differential equations is 8.

[2] Unless otherwise specified, index k may take the values 1, 2, . . . , N; while indices i and j take the values 1, 2.

First-Order Timoshenko-Type Model

In the second part of his *Course in Elasticity Theory* published in 1916, Timoshenko pointed out that it is necessary to take into consideration the transverse shear strain in the bending of beams. He introduced a correction to the curvature of the rod axis due to the shearing force. This theory is normally referred to as the Timoshenko-type theory.

This theory is based on the kinematic Timoshenko hypothesis

$$u_i^{(k)} = u_i + z\beta_i \qquad u_3^{(k)} = w \tag{5}$$

where β_i is the rotation component for the shell. The kinematic Timoshenko hypothesis implies that the straight line elements, normal to the undeformed reference surface of the shell, remain straight but not normal to the deformed reference surface and retain their original length. Also, it is assumed that the static hypothesis (condition in Eq 3) remains valid and the distribution of the transverse shear stresses $\sigma_{i3}^{(k)}$ over the thickness of the shell can be described by

$$\sigma_{i3}^{(k)} = f(z)\mu_i + N^-(z)p_i^- + N^+(z)p_i^+ \tag{6}$$

$$N^-(z) = (\delta_N - z)/h; \quad N^+(z) = (z - \delta_0)/h; \quad f(z) = 6N^-(z)N^+(z)/h$$

where p_i^- and p_i^+ are the intensities of the external loading acting on the bottom and top surfaces of the shell in the α_i coordinate directions; δ_0 and δ_N are the distances from the reference surface to the bottom and top surfaces of the shell; h is the total thickness of the shell; $N^-(z)$ and $N^+(z)$ are the shape functions of the shell; μ_i is the unknown function depending on α_1 and α_2. These functions can be expressed in terms of the kinematic variables u_i, w, and β_i by using the three-field Reissner mixed variational principle.

The feature of the Timoshenko-type model is that the relations of the generalized Hooke's law for transverse shear strains and stresses are satisfied integrally over the thickness of the shell (compared to the Kirchhoff-Love model where relations are discarded). However, in this model, the N generalized Hooke's equations for transverse normal strains (see again for comparison the Kirchhoff-Love model) are dropped. It should be noted that this requirement is not of principal character because an account of the transverse compression effect, which is quite important in solving contact problems, permits us to satisfy exactly all aforementioned generalized Hooke's equations for each layer. In that case, the transverse normal displacements $u_3^{(k)}$ are distributed over the thickness of the shell according to the nonlinear law. Here, we will develop this improved approach.

Note that the order of the governing system of the first-order Timoshenko-type theory equals 10.

Higher-Order Timoshenko-Type Model

It is well-known that the first-order Timoshenko-type theory leaves out the nonlinear dependence of strains on the thickness coordinate z. Knowledge of this fact makes nonlinear dependence important for the estimation of the stress-strain state at the radial tire belt edge.

The higher-order Timoshenko-type theory is based on the generalized Timoshenko hypothesis [29]:

$$u_i^{(k)} = u_i + z\theta_i + g(z)\psi_i \qquad u_3^{(k)} = w \qquad g(z) = \int_0^z f(z)dz \qquad (7)$$

where $g(z)$ is the continuous function of z characterizing the nonlinear thickness distribution of tangential displacements; θ_i and $f(z)$ are the functions defined in Eqs 4 and 6, respectively. It is also assumed that the static hypotheses (condition in Eq 3 and Eq 6) remain valid. As in the previous shell model, the relations of the generalized Hooke's law for transverse shear strains and stresses are satisfied integrally over the thickness of the shell.

Here, the order of the governing system of differential equations is 12.

First-Order Discrete-Layer Model

The first-order discrete-layer theory is based on a piecewise linear approximation for the tangential displacements in the thickness direction. This approximation was introduced by Grigolyuk in 1957 [27,28] and is usually called the kinematic Grigolyuk hypothesis. In this theory, the order of the governing equations is dependent on the number of layers of the shell and equals $4N + 6$.

If the reference surface of the shell is chosen to be the bottom surface (*i.e.*, $\delta_0 = 0$), the thickness variation of the displacements can be expressed in the following form:

$$u_i^{(k)} = u_i + \sum_{n=1}^{k-1} h_n\beta_i^{(n)} + (z - \delta_{k-1})\beta_i^{(k)} \qquad u_3^{(k)} = w \qquad (8)$$

where $\beta_i^{(k)}$ are the rotation components for the kth layer; h_k is the thickness of the kth layer; δ_k is the distance from the reference surface to the top surface of the kth layer.

The equations of this discrete-layer theory of multilayered anisotropic shells can be derived by adopting the traditional static hypothesis (condition in Eq 3) and a new static one

$$\sigma_{i3}^{(k)} = f_k(z)\mu_i^{(k)} + N_k^-(z)\tau_i^{(k-1)} + N_k^+(z)\tau_i^{(k)}$$
$$N_k^-(z) = (\delta_k - z)/h_k; \quad N_k^+(z) = (z - \delta_{k-1})/h_k; \quad f_k(z) = 6N_k^-(z)N_k^+(z)/h_k \qquad (9)$$

where $\tau_i^{(k-1)}$ and $\tau_i^{(k)}$ are the transverse shear stresses of the bottom and top surfaces of the kth layer; $N_k^-(z)$ and $N_k^+(z)$ are the shape functions of the kth layer. The functions $\tau_i^{(1)}, \ldots, \tau_i^{(N-1)}$, $\mu_i^{(k)}$ are unknown functions depending on the curvilinear coordinates α_1 and α_2 and can be expressed in terms of the kinematic variables u_i, w, and $\beta_i^{(k)}$ from the generalized Hooke's equations, which are satisfied integrally over the thickness of each layer. Obviously, Eq 9 generalizes Eq 6 and they coincide if $N = 1$.

Higher-Order Discrete-Layer Model

The higher-order discrete-layer model (HDM) is based on the generalized Grigolyuk hypothesis [25]:

$$u_i^{(k)} = u_i + z\theta_i + \sum_{n=1}^{k-1} (\zeta_n - \zeta_{n-1})\psi_i^{(n)} + [g(z) - \zeta_{k-1}]\psi_i^{(k)} \qquad u_3^{(k)} = w \quad (10)$$

where $\zeta_n = g(\delta_n)$, $(n = 0, 1, \ldots, N - 1)$; θ_i and $g(z)$ are the functions defined in Eq 7. The kinematic hypothesis (Eq 10) allows us to describe the nonlinear dependence of tangential stresses and strains on the thickness coordinate z for each layer and has enough general character. So, substituting $g(z) = z$ and $\psi_i^{(k)} = \beta_i^{(k)} - \theta_i$ into Eq 10, we get Eq 8. If we additionally assume that $\beta_i^{(k)} = \beta_i$ in Eq 8, we will arrive at Eq 5. Putting $\psi_i^{(k)} = \psi_i$ and $\zeta_0 = 0$ in Eq 10, we obtain Eq 7. Also, if we assume $\psi_i = 0$ in Eq 7, we will arrive at Eq 4. A geometric illustration of the kinematic hypotheses is given in Fig. 2.

Also, the equations of the higher-order discrete-layer theory of multilayered anisotropic shells can be derived by adopting the static hypotheses (condition in Eq 3 and Eq 9) and then using the Reissner mixed variational principle. This results in the relations of the generalized Hooke's law for transverse shear strains and stresses being satisfied integrally over the thickness of each layer. The order of the governing system of differential equations is also dependent on the number of layers of the shell and equals $4N + 8$.

Mixed Variational Equation for Geometrically Nonlinear Multilayered Anisotropic Shells

Consider the shell built up in the general case by the arbitrary superposition across the wall thickness of N thin layers of constant thickness h_k, where $k = 1, 2, \ldots, N$. The shell may be defined as a three-dimensional body of volume V bounded by two surfaces S^- and S^+, located at the distance δ_0 and δ_N with respect to the reference surface S and the edge boundary surface $\Omega = \Omega_1^- + \Omega_1^+ + \Omega_2^- + \Omega_2^+$, generated by the normals to the reference surface along the bounding curve $\Gamma = \Gamma_1^- + \Gamma_1^+ + \Gamma_2^- + \Gamma_2^+$. One of these edge boundary surfaces (*e.g.*, Ω_1^-) is defined by a set of points: $\alpha_1^- \leq \alpha_1 \leq \alpha_1^+$, $\alpha_2 = \alpha_2^-$, and $\delta_0 \leq z \leq \delta_N$, and the corresponding bounding curve Γ_1^- is defined by $\alpha_1^- \leq \alpha_1 \leq \alpha_1^+$, $\alpha_2 = \alpha_2^-$, and $z = 0$. It is also assumed that the bounding

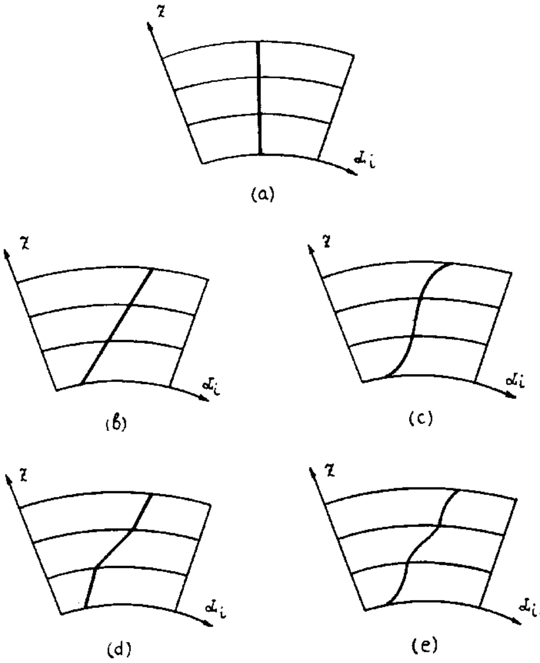

FIG. 2—*Distribution of the tangential displacements $u_i^{(k)}$ over the thickness of the three-layer shell; (a) Kirchhoff-Love model, (b) first-order Timoshenko-type model, (c) higher-order Timoshenko-type model, (d) first-order discrete-layer model, (e) higher-order discrete-layer model.*

surfaces and the reference surface are continuous, sufficiently smooth, and without any singularities.

Elastic Potential

Each layer is considered to be linearly elastic, homogeneous, and anisotropic (*i.e.*, in each point of the shell there is a single surface of elastic symmetry parallel to the reference surface). In this case, the elastic potential of the kth layer W_k can be written as follows:

$$
\begin{aligned}
W_k = {} & \tfrac{1}{2} a_{11}^{(k)}(\sigma_{11}^{(k)})^2 + a_{12}^{(k)}\sigma_{11}^{(k)}\sigma_{22}^{(k)} + a_{13}^{(k)}\sigma_{11}^{(k)}\sigma_{33}^{(k)} + a_{16}^{(k)}\sigma_{11}^{(k)}\sigma_{12}^{(k)} + \tfrac{1}{2} a_{22}^{(k)}(\sigma_{22}^{(k)})^2 \\
& + a_{23}^{(k)}\sigma_{22}^{(k)}\sigma_{33}^{(k)} + a_{26}^{(k)}\sigma_{22}^{(k)}\sigma_{12}^{(k)} + \tfrac{1}{2} a_{33}^{(k)}(\sigma_{33}^{(k)})^2 + a_{36}^{(k)}\sigma_{33}^{(k)}\sigma_{12}^{(k)} \\
& + \tfrac{1}{2} a_{66}^{(k)}(\sigma_{12}^{(k)})^2 + \tfrac{1}{2} a_{44}^{(k)}(\sigma_{23}^{(k)})^2 + a_{45}^{(k)}\sigma_{23}^{(k)}\sigma_{13}^{(k)} + \tfrac{1}{2} a_{55}^{(k)}(\sigma_{13}^{(k)})^2
\end{aligned} \tag{11}
$$

where $a_{11}^{(k)}, \ldots, a_{66}^{(k)}$ are the compliances of the kth layer.

Strain-Displacement Relationships

The three-dimensional partially nonlinear Novozhilov's strain-displacement relationships in the Lagrange description have the form [*30*]:

$$\epsilon_{11}^{(k)} = \frac{1}{1 + k_1 Z}\left(\frac{1}{A_1}\frac{\partial u_1^{(k)}}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_2^{(k)} + k_1 u_3^{(k)}\right) + \tfrac{1}{2}\,(\theta_1^{(k)})^2$$

$$\epsilon_{12}^{(k)} = \frac{1}{1 + k_1 z}\left(\frac{1}{A_1}\frac{\partial u_2^{(k)}}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_1^{(k)}\right) + \frac{1}{1 + k_2 z}\left(\frac{1}{A_2}\frac{\partial u_1^{(k)}}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u_2^{(k)}\right) + \theta_1^{(k)}\theta_2^{(k)} \tag{12}$$

$$\epsilon_{13}^{(k)} = \frac{\partial u_1^{(k)}}{\partial z} + \theta_1^{(k)}; \qquad \epsilon_{33}^{(k)} = \frac{\partial u_3^{(k)}}{\partial z}$$

$$\theta_1^{(k)} = \frac{1}{1 + k_1 z}\left(\frac{1}{A_1}\frac{\partial u_3^{(k)}}{\partial \alpha_1} - k_1 u_1^{(k)}\right) \qquad (1 \rightleftharpoons 2)$$

where $k_i = 1/R_i$ is the principal curvature of the reference surface. In Eq 12 only those nonlinear geometrical terms that depend on $\theta_i^{(k)}$ are retained. All of the remaining nonlinear terms are rejected.

Reissner Mixed Variational Principle

The governing equations of the theory of multilayered anisotropic shells can be obtained by applying the three-field Reissner mixed variational principle. It allows to reduce the three-dimensional state of stress to an equivalent two-dimensional state of stress and can be expressed in the following form:

$$\delta(J - A) = 0 \tag{13}$$

where A is the work of the external loads; the functional J is expressed by

$$J = \iint_S \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} [\sigma_{11}^{(k)}\epsilon_{11}^{(k)} + \sigma_{22}^{(k)}\epsilon_{22}^{(k)} + \sigma_{12}^{(k)}\epsilon_{12}^{(k)} + \sigma_{33}^{(k)}\epsilon_{33}^{(k)} + \sigma_{13}^{(k)}\epsilon_{13}^{(k)} + \sigma_{23}^{(k)}\epsilon_{23}^{(k)} - W_k]A_1A_2(1 + k_1 z)(1 + k_2 z)dz d\alpha_1 d\alpha_2 \tag{14}$$

Note that the fundamental equations of the nonlinear elasticity theory for the multilayered anisotropic shell result from the stationary condition in Eq 13, allowing for Eqs 11 and 12, for the independent and arbitrary variations of stresses and displacements within and on the boundary of the body.

Substituting the elastic potential W_k from Eq 11 and strain-displacement relationships from Eq 12 into Eqs 13 and 14, and using the generalized formulas of integration by parts, we will arrive at a mixed variational equation for multilayered anisotropic shells:

$$
\begin{aligned}
&\iint_S \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \{-L_1^{(k)}\delta u_1^{(k)} - L_2^{(k)}\delta u_2^{(k)} - L_3^{(k)}\delta u_3^{(k)} + [(\epsilon_{11}^{(k)} - a_{11}^{(k)}\sigma_{11}^{(k)} - a_{12}^{(k)}\sigma_{22}^{(k)} \\
&- a_{13}^{(k)}\sigma_{33}^{(k)} - \alpha_{16}^{(k)}\sigma_{12}^{(k)})\delta\sigma_{11}^{(k)} + (\epsilon_{22}^{(k)} - a_{12}^{(k)}\sigma_{11}^{(k)} - a_{22}^{(k)}\sigma_{22}^{(k)} - a_{23}^{(k)}\sigma_{33}^{(k)} - \alpha_{26}^{(k)}\sigma_{12}^{(k)})\delta\sigma_{22}^{(k)} \\
&+ (\epsilon_{33}^{(k)} - a_{13}^{(k)}\sigma_{11}^{(k)} - a_{23}^{(k)}\sigma_{22}^{(k)} - a_{33}^{(k)}\sigma_{33}^{(k)} - a_{36}^{(k)}\sigma_{12}^{(k)})\delta\sigma_{33}^{(k)} + (\epsilon_{12}^{(k)} - a_{16}^{(k)}\sigma_{11}^{(k)} \\
&- a_{26}^{(k)}\sigma_{22}^{(k)} - a_{36}^{(k)}\sigma_{33}^{(k)} - a_{66}^{(k)}\sigma_{12}^{(k)})\delta\sigma_{12}^{(k)} + (\epsilon_{23}^{(k)} - a_{44}^{(k)}\sigma_{23}^{(k)} - a_{45}^{(k)}\sigma_{13}^{(k)})\delta\sigma_{23}^{(k)} \\
&+ (\epsilon_{13}^{(k)} - a_{45}^{(k)}\sigma_{23}^{(k)} - a_{55}^{(k)}\sigma_{13}^{(k)})\delta\sigma_{13}^{(k)}]H_1H_2\}dz\,d\alpha_1 d\alpha_2 + \iint_S [(\sigma_{13}^{(N)} - p_1^+)\delta u_1^{(N)} \\
&+ (\sigma_{23}^{(N)} - p_2^+)\delta u_2^{(N)} + (\sigma_{33}^{(N)} - q^+)\delta u_3^{(N)}]H_1H_2|_{z=\delta_N} d\alpha_1 d\alpha_2 - \iint_S [(\sigma_{13}^{(1)} \\
&- p_1^-)\delta u_1^{(1)} + (\sigma_{23}^{(1)} - p_2^-)\delta u_2^{(1)} + (\sigma_{33}^{(1)} - q^-)\delta u_3^{(1)}]H_1H_2|_{z=\delta_0} d\alpha_1 d\alpha_2 \\
&+ \iint_S \sum_{n=1}^{N-1} [(\sigma_{13}^{(n)}\delta u_1^{(n)} - \sigma_{13}^{(n+1)}\delta u_1^{(n+1)}) + (\sigma_{23}^{(n)}\delta u_2^{(n)} - \sigma_{23}^{(n+1)}\delta u_2^{(n+1)}) \qquad (15) \\
&+ (\sigma_{33}^{(n)}\delta u_3^{(n)} - \sigma_{33}^{(n+1)}\delta u_3^{(n+1)})]\, H_1H_2|_{z=\delta_n} d\alpha_1 d\alpha_2 + \int_{\Gamma_2^+} \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \\
&\times [(\sigma_{11}^{(k)} - X_1^+)\delta u_1^{(k)} + (\sigma_{12}^{(k)} - X_2^+)\delta u_2^{(k)} + (S_{13}^{(k)} - X_3^+)\delta u_3^{(k)}]H_2|_{\alpha_1=\alpha_1^+} dz\,d\alpha_2 \\
&- \int_{\Gamma_2^-} \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} [(\sigma_{11}^{(k)} - X_1^-)\delta u_1^{(k)} + (\sigma_{12}^{(k)} - X_2^-)\delta u_2^{(k)} + (S_{13}^{(k)} - X_3^-\delta u_3^{(k)}] \\
&\times H_2|_{\alpha_1=\alpha_1^-} dz\,d\alpha_2 + \int_{\Gamma_1^+} \sum_{k=1}^{N} \int_{\delta_{k=1}}^{\delta_k} [(\sigma_{12}^{(k)} - Y_1^+)\delta u_1^{(k)} + (\sigma_{22}^{(k)} - Y_2^+)\delta u_2^{(k)} \\
&+ (S_{23}^{(k)} - Y_3^+)\delta u_3^{(k)}]H_1|_{\alpha_2=\alpha_2^+} dz\,d\alpha_1 - \int_{\Gamma_1^-} \sum_{k=1}^{N} \int_{\delta k-1}^{\delta_k} [(\sigma_{12}^{(k)} - Y_1^-)\delta u_1^{(k)} \\
&+ (\sigma_{22}^{(k)} - Y_2^-)\delta u_2^{(k)} + (S_{23}^{(k)} - Y_3^-)\delta u_3^{(k)}]H_1|_{\alpha_2=\alpha_2^-} dz\,d\alpha_1 = 0
\end{aligned}
$$

where p_i^-, q^- and p_i^+, q^+ are the intensities of the external loading, acting on the bottom surface S^- and top surface S^+ of the shell, in the α_i and z coordinate directions, respectively; X_i^-, X_3^- and X_i^+, X_3^+ are the intensities of the external loading, acting on the edge boundary surfaces Ω_2^- and Ω_2^+, in the α_i and z coordinate directions; Y_i^-, Y_3^- and Y_i^+, Y_3^+ are the intensities of the external loading, acting on the edge boundary surfaces Ω_1^- and Ω_1^+, in the α_i and z coordinate directions; $H_i = A_i(1 + k_i z)$ is the Lamé coefficients; L_i and L_3 are the three-dimensional nonlinear differential operators, corresponding to the adopting Novozhilov's strain-displacement relationships, which can be written as follows [*30*]:

$$L_1^{(k)} = \frac{\partial(H_2\sigma_{11}^{(k)})}{\partial\alpha_1} - \frac{H_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\sigma_{22}^{(k)} + \frac{\partial(H_1\sigma_{12}^{(k)})}{\partial\alpha_2}$$

$$+ \frac{H_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}\sigma_{12}^{(k)} + \frac{\partial(H_1H_2\sigma_{13}^{(k)})}{\partial z} + k_1A_1H_2S_{13}^{(k)} \tag{16}$$

$$L_3^{(k)} = \frac{\partial(\mathrm{H}_2S_{13}^{(k)})}{\partial\alpha_1} + \frac{\partial(H_1S_{23}^{(k)})}{\partial\alpha_2} + \frac{\partial(H_1H_2\sigma_{33}^{(k)})}{\partial z} - k_1A_1H_2\sigma_{11}^{(k)} - k_2A_2H_1\sigma_{22}^{(k)}$$

$$S_{13}^{(k)} = \sigma_{13}^{(k)} + \theta_1^{(k)}\sigma_{11}^{(k)} + \theta_2^{(k)}\sigma_{12}^{(k)}\,(1 \rightleftharpoons 2)$$

With the help of this mixed variational equation, we can construct any nonlinear theory of multilayered anisotropic shells. Here, for conciseness, we will consider only one theory, namely, the nonlinear first-order discrete-layer theory, which will demonstrate the feature of our approach.

Nonlinear First-Order Discrete-Layer Theory of Multilayered Anisotropic Shells

Basic Assumptions

The equations of the first-order discrete-layer theory of thin multilayered anisotropic shells will be derived by adopting the following basic assumptions:

1. The layers constituting the shell are rigidly joined so that no slip occurs on the contact surfaces.
2. The material of each component layer of the shell is assumed to be linearly elastic, homogeneous, and anisotropic so that the elastic potential is defined by Eq 11.
3. The well-known nonlinear Novozhilov's strain-displacement relationships in Eq 12 are invoked.
4. The reference surface of the shell is chosen to be the bottom surface (*i.e.*, $S^- = S$).
5. The tangential displacements are distributed over the thickness of the shell according to the kinematic Grigolyuk hypothesis (Eq 8). For convenience, we rewrite one in the following for:

$$u_i^{(k)} = u_i + \sum_{n=1}^{N} \pi_{kn}\beta_i^{(n)} + (z - \delta_{k-1})\beta_i^{(k)}$$

$$\pi_{kn} = \begin{cases} h_n & \text{if } k > n \\ 0 & \text{if } k \leq n \end{cases} \tag{17}$$

6. The transverse displacements are distributed over the thickness of the shell according to the nonlinear law

$$u_3^{(k)} = w + \gamma^{(k)} \tag{18}$$

where $\gamma^{(k)}$ is an unknown function depending on the external loads only and slowly varying in the coordinate direction α_i, hence their derivatives with respect to α_i are considered negligibly small.
7. The transverse shear stresses are distributed over the thickness of the shell according to Eq 9.

It should be mentioned that the first-order discrete-layer theory based on the simplest approximation (uniform distribution) for transverse displacement was developed in [*23*]. Later, in [*24*] the piecewise linear approximation was used for transverse displacements. Here, we will consider the more general case (Eq 18).

Basis Equations

Substituting the displacements from Eqs 17 and 18 and the transverse shear stresses from Eq 9 into the mixed variational equation (Eq 15), and equating to zero the factors at arbitrary variations of the independent variables u_i, w, $\beta_i^{(k)}$, $\sigma_{ij}^{(k)}$, $\sigma_{33}^{(k)}$, $\mu_i^{(k)}$, and $\tau_i^{(1)}, \ldots, \tau_i^{(N-1)}$ (allowing for the fact that $\partial\gamma^{(k)} = 0$), we obtained the following original relationships of the thin shell theory:

1. The physical equations of the generalized Hooke's law:

$$\begin{aligned}
\epsilon_{11}^{(k)} &= a_{11}^{(k)}\sigma_{11}^{(k)} + a_{12}^{(k)}\sigma_{22}^{(k)} + a_{13}^{(k)}\sigma_{33}^{(k)} + a_{16}^{(k)}\sigma_{12}^{(k)} \\
\epsilon_{22}^{(k)} &= a_{12}^{(k)}\sigma_{11}^{(k)} + a_{22}^{(k)}\sigma_{22}^{(k)} + a_{23}^{(k)}\sigma_{33}^{(k)} + a_{26}^{(k)}\sigma_{12}^{(k)} \\
\epsilon_{33}^{(k)} &= a_{13}^{(k)}\sigma_{11}^{(k)} + a_{23}^{(k)}\sigma_{22}^{(k)} + a_{33}^{(k)}\sigma_{33}^{(k)} + a_{36}^{(k)}\sigma_{12}^{(k)} \\
\epsilon_{12}^{(k)} &= a_{16}^{(k)}\sigma_{11}^{(k)} + a_{26}^{(k)}\sigma_{22}^{(k)} + a_{36}^{(k)}\sigma_{33}^{(k)} + a_{66}^{(k)}\sigma_{12}^{(k)}
\end{aligned} \tag{19}$$

2. The integral physical equations of the generalized Hooke's law for transverse shear strains and stresses:

$$\int_{\delta k-1}^{\delta_k} \Lambda_i^{(k)} f_k(z)dz = 0$$

$$\int_{\delta_{n-1}}^{\delta_n} \Lambda_i^{(n)} N_n^+(z)dz + \int_{\delta n}^{\delta n+1} \Lambda_i^{(n+1)} N_{n+1}^-(z)dz = 0 \tag{20}$$

$$(n = 1, 2, \ldots, N - 1)$$

$$\Lambda_1^{(k)} = \epsilon_{13}^{(k)} - a_{45}^{(k)}\sigma_{23}^{(k)} - a_{55}^{(k)}\sigma_{13}^{(k)} \qquad (1 \leftrightarrows 2, 4 \leftrightarrows 5)$$

3. $2N + 2$ equilibrium equations of the shell in terms of the stress and moment resultants:

$$\sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} L_i^{(k)} dz = 0$$
$$\int_{\delta_{k-1}}^{\delta_k} L_i^{(k)}(z - \delta_{k-1}) dz + \sum_{n=1}^{N} \pi_{nk} \int_{\delta_n}^{\delta_n} L_i^{(n)} dz = 0 \tag{21}$$

4. N nonlinear equations of the three-dimensional elasticity theory:

$$L_3^{(k)} = 0 \tag{22}$$

5. The boundary conditions for the transverse stresses on the bottom surface $z = \delta_0$:

$$\sigma_{i3}^{(1)} = p_i^- \qquad \sigma_{33}^{(1)} = q^- \tag{23}$$

the first two of which are identically satisfied according to Eq 9.

6. The boundary conditions for the transverse stresses on the top surface $z = \delta_N$:

$$\sigma_{i3}^{(N)} = p_i^+ \qquad \sigma_{33}^{(N)} = q^+ \tag{24}$$

the first two of which are identically satisfied according to Eq 9.

7. The equilibrium conditions for the transverse stresses at the layer interfaces $z = \delta_n$:

$$\sigma_{i3}^{(n)} = \sigma_{i3}^{(n+1)} \qquad \sigma_{33}^{(n)} = \sigma_{33}^{(n+1)} \qquad (n = 1, 2, \ldots, N-1) \tag{25}$$

the first $2N - 2$ of which are identically satisfied according to Eq 9.

8. The natural boundary conditions for the stress and moment resultants and generalized displacements on the edge surfaces $\alpha_1 = \alpha_1^+$ and $\alpha_1 = \alpha_1^-$:

$$(T_{11} - T_{11}^{\pm})\delta u_1 = 0; \quad (T_{12} - T_{12}^{\pm})\delta u_2 = 0; \quad (N_1 - Q_1^{\pm})\delta w = 0$$
$$(\Phi_{11}^{(k)} - \Phi_{11}^{(k)\pm})\delta\beta_1^{(k)} = 0; \quad (\Phi_{12}^{(k)} - \Phi_{12}^{(k)\pm})\delta\beta_2^{(k)} = 0 \tag{26}$$

9. The natural boundary conditions for the stress and moment resultants and generalized displacements on the edge surfaces $\alpha_2 = \alpha_2^+$ and $\alpha_2 = \alpha_2^-$, which are correspondingly obtained from Eq 26 by replacing subscript 1 with 2 and vice versa.

In Eqs 21 and 26 the stress and moment resultants per unit length of the coordinate curves are defined by

$$T_{ij}^{(k)} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{ij}^{(k)} dz; \qquad M_{ij}^{(k)} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{ij}^{(k)} z dz; \qquad Q_i^{(k)} = \int_{\delta k-1}^{\delta_k} \sigma_{i3}^{(k)} dz$$

$$\Phi_{ij}^{(k)} = M_{ij}^{(k)} - \delta_{k-1} T_{ij}^{(k)} + \sum_{n=1}^{N} \pi_{nk} T_{ij}^{(n)} \tag{27}$$

$$T_{ij} = \sum_{k=1}^{N} T_{ij}^{(k)} \qquad Q_i = \sum_{k=1}^{N} Q_i^{(k)}$$

$$N_1 = Q_1 - \theta_1 T_{11} - \theta_2 T_{12} \quad \theta_1 = k_1 u_1 - A_1^{-1} \partial w / \partial \alpha_1 \quad (1 \rightleftharpoons 2)$$

Strain-Displacement Relationships

The substitution of Eqs 17 and 18 into Eq 12 yields the nonvanishing strain components at any point of the structure space as expressed by

$$\epsilon_{ij}^{(k)} = E_{ij} + \sum_{n=1}^{N} \pi_{kn} K_{ij}^{(n)} + (z - \delta_{k-1}) K_{ij}^{(k)} \tag{28}$$

$$\epsilon_{i3}^{(k)} = \beta_i^{(k)} - \theta_i; \qquad \epsilon_{33}^{(k)} = \partial \gamma^{(k)} / \partial z$$

where E_{11} and E_{22} are the extensional strains in the coordinate directions α_1 and α_2 of the reference surface points; E_{12} is the tangential shearing strain of the reference surface points; $K_{11}^{(k)}$, $K_{22}^{(k)}$, and $K_{12}^{(k)}$ are the functions characterizing the bending and twisting of the face surfaces of the kth layer, respectively:

$$E_{11} = \epsilon_1 + \theta_1^2/2 \qquad E_{12} = \omega_1 + \omega_2 + \theta_1 \theta_2$$

$$K_{11}^{(k)} = æ_1^{(k)} \qquad K_{12}^{(k)} = \xi_1^{(k)} + \xi_2^{(k)}$$

$$\epsilon_1 = \frac{1}{A_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_2 + k_1 w \quad \omega_1 = \frac{1}{A_1} \frac{\partial u_2}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_1 \tag{29}$$

$$æ_1^{(k)} = \frac{1}{A_1} \frac{\partial \beta_1^{(k)}}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \beta_2^{(k)} \quad \xi_1^{(k)} = \frac{1}{A_1} \frac{\partial \beta_2^{(k)}}{\partial \alpha_1} - \frac{1}{A_1 A_2} \beta_1^{(k)} \quad (1 \rightleftharpoons 2)$$

Equations 28 and 29 correspond to the thin shells having moderate rotations around the normal to the shell's reference surface. Obviously, this situation occurs in tire structural mechanics.

Equilibrium Equations of the Shell

Taking into account Eq 27, the equilibrium equations of the shell in Eq 21 are written as follows:

$$\frac{\partial(A_2T_{11})}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1}T_{22} + \frac{\partial(A_1T_{12})}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}T_{12} + A_1A_2k_1N_1$$

$$= A_1A_2(p_1^- - p_1^+) \qquad (1 \rightleftharpoons 2) \quad (30)$$

$$\frac{\partial(A_2\Phi_{11}^{(k)})}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1}\Phi_{22}^{(k)} + \frac{\partial(A_1\Phi_{12}^{(k)})}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}\Phi_{12}^{(k)} - A_1A_2Q_1^{(k)} = -A_1A_2h_kp_1^+ \quad (1 \rightleftharpoons 2)$$

By integrating the nonlinear equations of the three-dimensional elasticity theory (Eq 22) across the shell thickness and allowing for the third boundary condition defined by Eq 23 and equilibrium conditions for the transverse stresses at the layer interfaces (Eq 25), we obtain the following expression:

$$\sigma_{33}^{(k)} = q^- - \frac{1}{A_1A_2}\left(\frac{\partial(A_2N_1^{(k)}[z])}{\partial\alpha_1} + \frac{\partial(A_1N_2^{(k)}[z]}{\partial\alpha_2}\right) + k_1T_{11}^{(k)}[z] + k_2T_{22}^{(k)}[z]$$

$$T_{ij}^{(k)}[z] = \sum_{n=1}^{k-1} T_{ij}^{(n)} + \int_{\delta_{k-1}}^{z} \sigma_{ij}^{(k)}dz \qquad Q_i^{(k)}[z] = \sum_{n=1}^{k-1} Q_i^{(n)} + \int_{\delta_{k-1}}^{z} \sigma_{i3}^{(k)}dz \quad (31)$$

$$N_1^{(k)}[z] = Q_1^{(k)}[z] - \theta_1T_{11}^{(k)}[z] - \theta_2T_{12}^{(k)}[z] \qquad (1 \rightleftharpoons 2)$$

In conclusion, using the third boundary condition defined by Eq 24 and taking into consideration obvious equalities $T_{ij}^{(k)}[\delta_N] = T_{ij}$, $Q_i^{(N)}[\delta_N] = Q_i$, and $N_i^{(N)}[\delta_N] = N_i$, we arrive at the last equilibrium equation

$$\frac{\partial(A_2N_1)}{\partial\alpha_1} + \frac{\partial(A_1N_2)}{\partial\alpha_2} - A_1A_2(k_1T_{11} + k_2T_{22}) = A_1A_2(q^- - q^+) \qquad (32)$$

So, we have $2N + 3$ differential equations (see Eqs 30 and 32) describing the equilibrium of the multilayered anisotropic shell, the number of which corresponds to the number of the generalized displacements u_i, w, and $\beta_i^{(k)}$.

Constitutive Equations

For the purpose of simplifying the analysis further, we also adopt the static Kirchhoff-Love hypothesis (condition in Eq 3). As it has already been stated, this assumption is associated with small values of the transverse normal stresses $\sigma_{33}^{(k)}$ in thin-walled structures; therefore, the terms containing the stresses $\sigma_{33}^{(k)}$ in the first, second, and fourth terms in Eq 19 can be omitted. The inverted form of these simplified equations will be

$$\|\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \sigma_{12}^{(k)}\|^T = \mathbf{b}^{(k)} \cdot \|\epsilon_{11}^{(k)}, \epsilon_{22}^{(k)}, \epsilon_{12}^{(k)}\|^T \qquad (33)$$

where $\mathbf{b}^{(k)}$ is the stiffness matrix of the kth layer defined in [2]; the superscript T denotes transposition.

Replacing the tangential stresses from Eq 33 into Eq 27, and taking into account Eq 28, we obtain the following constitutive equations of the theory of thin multilayered anisotropic shells

$$\|T_{11}, T_{22}, T_{12}\|^T = \mathbf{A}\cdot\|E_{11}, E_{22}, E_{12}\|^T + \sum_{n=1}^{N} \mathbf{D}^{(n)}\cdot\|K_{11}^{(n)}, K_{22}^{(n)}, K_{12}^{(n)}\|^T$$
$$\|\Phi_{11}^{(k)}, \Phi_{22}^{(k)}, \Phi_{12}^{(k)}\|^T = \mathbf{D}^{(k)}\cdot\|E_{11}, E_{22}, E_{12}\|^T + \sum_{n=1}^{N} \mathbf{F}^{(kn)}\|K_{11}^{(n)}, K_{22}^{(n)}, K_{12}^{(n)}\|^T \quad (34)$$

where $\mathbf{A}$, $\mathbf{D}^{(k)}$, and $\mathbf{F}^{(kn)}$ are the shell stiffness matrices given in [*2*] and [*24*].

By introducing Eq 9 into the integral equations of the generalized Hooke's law in Eq 20, and excluding the functions $\mu_i^{(k)}$, we will find the system of linear algebraic equations for the interlaminar shear stresses

$$\mathbf{G}\cdot\|\tau_1^{(1)}, \ldots, \tau_1^{(N-1)}, \tau_2^{(1)}, \ldots, \tau_2^{(N-1)}\|^T = \mathbf{V} \quad (35)$$

The matrix $\mathbf{G}$ and vector $\mathbf{V}$ are defined in [*24*].

If we solve the linear system in Eq 35 by using the well-known numerical methods, we will find interlaminar stresses $\tau_i^{(1)}, \ldots, \tau_i^{(N-1)}$ and the function $\mu_i^{(k)}$ from Eq 20 will be expressed further in terms of the transverse shear strains and interlaminar stresses as follows:

$$\mu_1^{(k)} = q_{44}^{(k)}\epsilon_{13}^{(k)} - q_{45}^{(k)}\epsilon_{23}^{(k)} - \frac{5}{12} h_k(\tau_1^{(k-1)} + \tau_1^{(k)}) \qquad (1 \leftrightarrows 2; 4 \leftrightarrows 5) \quad (36)$$

where $q_{mn}^{(k)}$ is the transverse shear coefficient defined in [*24*].

Equation 20 is important to the proposed discrete-layer theory of multilayered anisotropic shells because it exhibits the connection of the interlaminar stresses $\tau_i^{(1)}, \ldots, \tau_i^{(N-1)}$ and functions $\mu_i^{(k)}$ with the generalized displacements u_i, w, and $\beta_i^{(k)}$.

Finally, substituting Eq 9 into Eq 27, and taking into account Eq 36, we obtain the expression for the transverse shear stress resultants as follows:

$$Q_1^{(k)} = q_{44}^{(k)}\epsilon_{13}^{(k)} - q_{45}^{(k)}\epsilon_{23}^{(k)} + \frac{1}{12} h_k(\tau_1^{(k-1)} + \tau_1^{(k)}) \qquad (1 \leftrightarrows 2; 4 \leftrightarrows 5) \quad (37)$$

Computation of Transverse Compression

Now, we can find the transverse normal stresses $\sigma_{33}^{(k)}$ from Eq 31. Replacing their values and tangential stresses $\sigma_{ij}^{(k)}$ into the third term of the generalized Hooke's law (Eq 19) yields the required formula for the transverse normal strains $\epsilon_{33}^{(k)}$.

To find the functions $\gamma^{(k)}$ in Eq 18 characterizing a nonlinear distribution of the transverse displacements over the thickness of the shell, we should integrate the last group of Eq 28 across the shell thickness and take into account the continuity conditions for the displacements $\gamma^{(n)}(\delta_n) = \gamma^{(n+1)}(\delta_n)$, where $n = 1, 2, \ldots, N-1$. The result is

$$\gamma^{(k)} = \sum_{n=1}^{k-1} \int_{\delta_{n-1}}^{\delta_n} \epsilon_{33}^{(n)} dz + \int_{\delta_{k-1}}^{z} \epsilon_{33}^{(k)} dz$$

With the computation of the transverse displacements $u_3^{(k)}$ in Eq 18, the general description of the refined theory in [*13*] and [*24*] is completed.

Axisymmetric Deformation of Multilayered Anisotropic Shells of Revolution

In this section, multilayered anisotropic shells of revolution with uniform circumferential properties subjected to axisymmetric loading are considered. In this case, the shell will deform axisymmetrically, always remaining as a body of revolution, and the generalized displacements u_i, w, and $\beta_i^{(k)}$ will depend only on the meridional coordinate α_1.

Let $\mathbf{Y}$ be the resolution vector given by

$$\begin{aligned} \mathbf{Y} = \|&T_{11}, N_1, \Phi_{11}^{(1)}, \ldots, \Phi_{11}^{(N)}, T_{12}, \\ &\Phi_{12}^{(1)}, \ldots, \Phi_{12}^{(N)}, u_1, w, \\ &\beta_1^{(1)}, \ldots, \beta_1^{(N)}, u_2, \beta_2^{(1)}, \ldots, \beta_2^{(N)}\|^T \end{aligned} \tag{38}$$

With the help of $\mathbf{Y}$, we can rewrite the nonlinear system of governing equations in the following vector form

$$A_1^{-1}\, d\mathbf{Y}/d\alpha_1 = \mathbf{F}(\alpha_1, \mathbf{Y}) \tag{39}$$

where vector $\mathbf{F}$ is the right side of the system defined by Eqs 27, 29, 30, and 32. As it has already been noted, the order of the governing system in Eq 39 is dependent on the number of layers of the shell and equals $4N + 6$.

The boundary conditions of the axisymmetric problem can be written as follows:

$$\begin{aligned} Y_m(\alpha_1^-)l_m + Y_{2N+3+m}(\alpha_1^-)(1 - l_m) &= 0 \\ Y_m(\alpha_1^+)1_{2N+3+m} + Y_{2N+3+m}(\alpha_1^+)(1 - l_{2n+3+m}) &= 0 \end{aligned} \tag{40}$$

where Y_m and Y_{2N+3+m} are the components of the resolution vector; l_m and l_{2N+3+m} are the boundary coefficients, which may take the values 0 and 1 and define any homogeneous static or kinematic boundary conditions at the left and right edges of the shell; $m = 1, 2, \ldots, 2N + 3$.

The nonlinear boundary problem in Eqs 39 and 40 can be reduced to a sequence of linear boundary problems by applying the modified Newton method. The linear boundary problem is solved through the use of the Godunov method of numerical integration [2].

Nonaxisymmetric Deformation of Multilayered Anisotropic Shells of Revolution

Let the multilayered anisotropic shell of revolution be subjected first to *axisymmetric loads* p_i^{-0}, p_i^{+0}, q^{-0}, and q^{+0}. In this case, the shell will deform axisymmetrically and its stress-strain state, which we will call basic, is characterized by two vectors:

$$\mathbf{X}_1^0(\alpha_1) = \|T_{11}^0, T_{22}^0, N_1^0, Q_1^{(1)0}, \ldots, Q_1^{(N)0}, \\ \Phi_{11}^{(1)0}, \ldots, \Phi_{11}^{(N)0}, \Phi_{22}^{(1)0}, \ldots, \Phi_{22}^{(N)0}, u_1^0, w^0, \beta_1^{(1)0}, \ldots, \beta_1^{(N)0}\|^T \tag{41}$$

$$\mathbf{X}_2^0(\alpha_1) = \|T_{12}^0, N_2^0, Q_2^{(1)0}, \ldots, Q_2^{(N)0}, \Phi_{12}^{(1)0}, \ldots, \Phi_{12}^{(N)0}, u_2^0, \beta_2^{(1)0}, \ldots, \beta_2^{(N)0}\|^T$$

Then, the shell is subjected to nonaxisymmetric loads p_i^-, p_i^+, q^-, and q^+. A new stress-strain state is characterized by vectors with the superscript Σ:

$$\mathbf{X}_i^\Sigma\ (\alpha_1, \alpha_2) = \mathbf{X}_i^0(\alpha_1) + \mathbf{X}_i(\alpha_1, \alpha_2) \tag{42}$$

where $\mathbf{X}_i(\alpha_1, \alpha_2)$ are the vectors characterizing the additional stress-strain state, which is near the basic one.

Let us suppose that vectors $\mathbf{X}_i(\alpha_1, \alpha_2)$ and loading components p_i^-, p_i^+, q^-, and q^+ are the periodical functions from the circumferential coordinate α_2, which can be expanded in the Fourier series in this coordinate

$$\mathbf{X}_1(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} [\mathbf{X}_{1,n}(\alpha_1) \cos n\alpha_2 + \mathbf{X}_{1,-n}(\alpha_1) \sin n\alpha_2]$$

$$\mathbf{X}_2(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} [\mathbf{X}_{2,n}(\alpha_1) \sin n\alpha_2 + \mathbf{X}_{2,-n}(\alpha_1) \cos n\alpha_2]$$

$$p_1^{\pm}(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} [p_{1,n}^{\pm}(\alpha_1) \cos n\alpha_2 + p_{1,-n}^{\pm}(\alpha_1) \sin n\alpha_2] \tag{43}$$

$$p_2^{\pm}(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} [p_{2,n}^{\pm}(\alpha_1) \sin n\alpha_2 + p_{2,-n}^{\pm}(\alpha_1) \cos n\alpha_2]$$

$$q^{\pm}(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} [q_n^{\pm}(\alpha_1) \cos n\alpha_2 + q_{-n}^{\pm}(\alpha_1) \sin n\alpha_2]$$

Replacing the components of vectors $\mathbf{X}_i^\Sigma(\alpha_1, \alpha_2)$ and the external loads from Eqs 42 and 43 into Eqs 27, 29, 30, and 32, and taking into account Eq 39 describing the basic stress-strain state, we will arrive at the governing equations describing the additional stress-strain state:

$$A_1^{-1}\ d\mathbf{Z}_n/d\alpha_1 = \mathbf{R}_n(\alpha_1, \mathbf{Z}_n) \qquad (n = 1, 2, \ldots) \tag{44}$$

where $\mathbf{Z}_n$ is the resolution vector corresponding to the nth Fourier harmonic defined by

$$\mathbf{Z}_n = \|T_{11,n}, T_{11,-n}, N_{1,n}, N_{1,-n}, \Phi_{11,n}^{(1)}, \ldots, \Phi_{11,n}^{(N)}, \Phi_{11,-n}^{(1)}, \ldots, \Phi_{11,-n}^{(N)}, \\ T_{12,n}, T_{12,-n}, \Phi_{12,n}^{(1)}, \ldots, \Phi_{12,n}^{(N)}, \Phi_{12,-n}^{(1)}, \ldots, \Phi_{12,-n}^{(N)}, \\ u_{1,n}, u_{1,-n}, w_n, w_{-n}, \beta_{1,n}^{(1)}, \ldots, \beta_{1,n}^{(N)}, \beta_{1,-n}^{(1)}, \ldots, \beta_{1,-n}^{(N)}, \\ u_{2,n}, u_{2,-n}, \beta_{2,n}^{(1)}, \ldots, \beta_{2,n}^{(N)}, \beta_{2,-n}^{(1)}, \ldots, \beta_{2,-n}^{(N)}\|^T \tag{45}$$

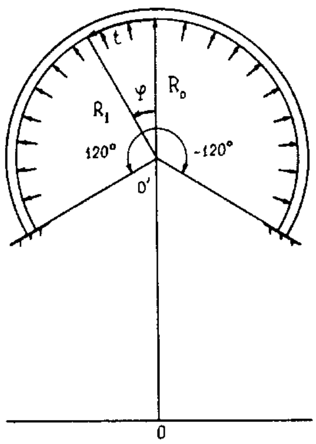

FIG. 3 — *Four-layer anisotropic tire subjected to inflation pressure.*

Note that according to Eq 42 in Eqs 44 and 45, all nonlinear terms must be omitted. Also, the order of the governing system equals $8N + 12$ because for anisotropic shells of revolution, the two sets of symmetric and antisymmetric generalized displacements and stress resultants, associated with each Fourier harmonic, are coupled.

The boundary conditions of the nonaxisymmetric problem can be written as follows:

$$\begin{aligned} &Z_{m,n}(\alpha_1^-)l_m + Z_{4N+6+m,n}(\alpha_1^-)(1 - l_m) = 0 \\ &Z_{m,n}(\alpha_1^+)l_{4N+6+m} + Z_{4N+6+m,n}(\alpha_1^+)(1 - l_{4N+6+m}) = 0 \quad (n = 1, 2 \ldots) \end{aligned} \tag{46}$$

where $Z_{m,n}$ and $Z_{4N+6+m,n}$ are the components of the resolution vector $\mathbf{Z}_n$; l_m and l_{4N+6+m} are the boundary coefficients, which may take the values 0 and 1; $m = 1, 2, \ldots, 4N + 6$.

As a result, the two-dimensional boundary problem is reduced to a sequence of the single-dimensional linear boundary problems in Eqs 44 and 46, which can be solved by using the above-mentioned Godunov's method.

Numerical Results

In this section, several numerical examples are presented. These examples include some relatively simple problems, namely: the nonlinear axisymmetric response of an anisotropic tire subjected to inflation pressure and the nonaxisymmetric response of an anisotropic tire subjected to the normal inflation

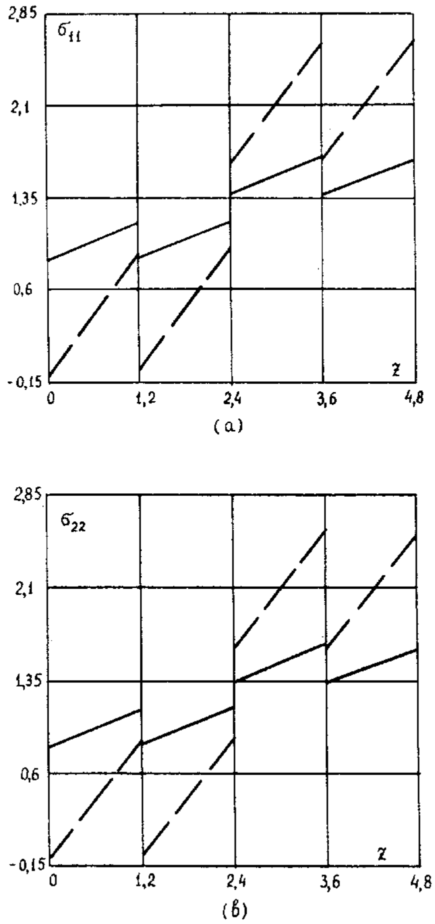

FIG. 4 — *Distribution of the stresses σ_{11} and σ_{22} in the thickness direction at the cross section (φ = 60°); nonlinear (——) and linear (— —) solutions.*

pressure and localized loading. For the sake of simplicity, the tire is modeled by a four-layer anisotropic toroidal shell, which has a circular cross section.

Axisymmetric Deformation of an Anisotropic Tire Subjected to Inflation Pressure

Let a four-layer anisotropic toroidal shell be subjected to a uniform inflation pressure of 0.15 MPa (Fig. 3). The material characteristics of the layers were taken to be those typical of cord-rubber composites [2]: E_L = 510.45

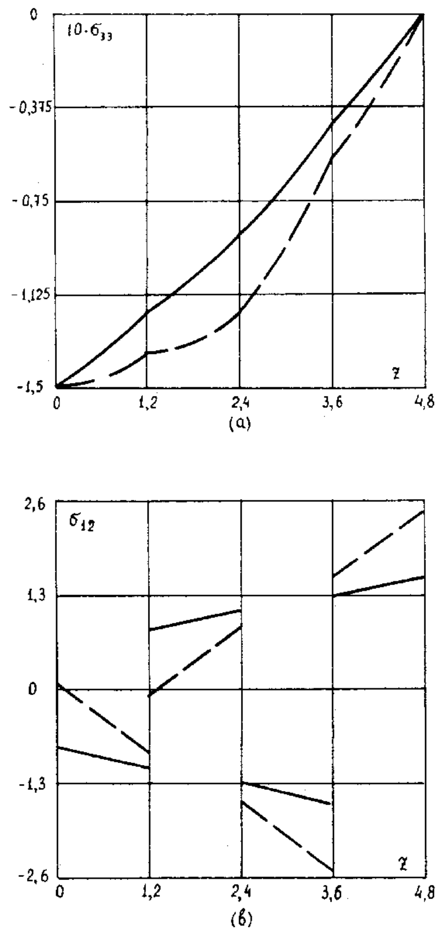

FIG. 5 — *Distribution of the stresses σ_{33} and σ_{12} in the thickness direction at the cross section (φ = 60°); nonlinear (——) and linear (— —) solutions.*

MPa, E_T = 6.91 MPa, $G_{LT} = G_{LZ}$ = 2.33 MPa, G_{TZ} = 1.77 MPa, ν_{LT} = 0.46, and ν_{TZ} = 0.95, where the subscripts L, T, and Z refer to longitudinal, transverse, and thickness directions of the individual ply; ν_{LT} is the Poisson's ratio measuring strain in the T direction under uniaxial normal stress in the L direction. The geometrical characteristics of the inner surface of the shell are R_1 = 50 mm, R_0 = 250 mm, and h = 4.8 mm. It is assumed that ply orientations and thicknesses are $\gamma_k = (-1)^{k-1} \cdot 52°$ and h_k = 1.2 mm, where k = 1, 2, 3, 4. The shell is also assumed to be rigidly clamped at the rim (at $\varphi = \pm 120°$).

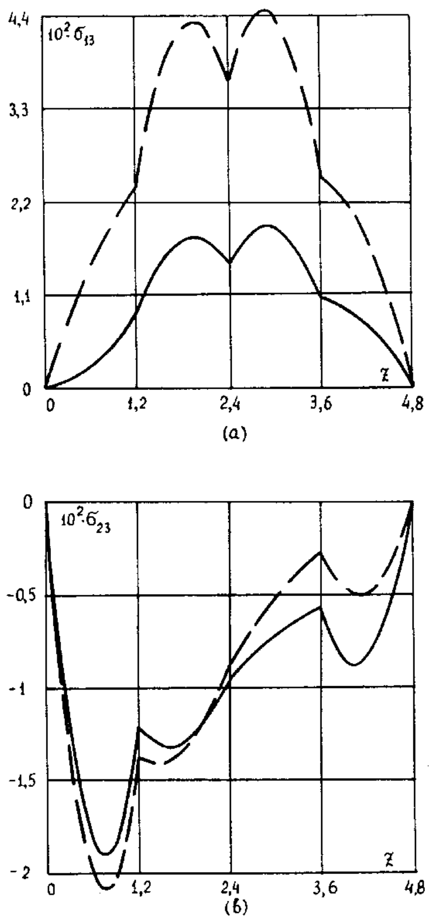

FIG. 6 — *Distribution of the stresses σ_{13} and σ_{23} in the thickness direction at the cross section* (φ = 60°); *nonlinear* (——) *and linear* (— —) *solutions.*

The numerical results presented in Figs. 4, 5, and 6 are obtained by integrating the governing system of nonlinear differential equations (Eq 39), the order of which equals 22. The distribution of the stresses in the thickness direction shown in Figs. 4, 5, and 6 is based on the nonlinear solution (solid lines) and linear solution (dash lines) at the cross section (at $\varphi = 60°$). Let's pay attention to the same order of the transverse shear stresses σ_{13} and σ_{23} (see Fig. 6); it is essentially noticeable, namely, for the nonlinear problem. It points to an essential influence of anisotropy and geometrical nonlinearity on the stress field even in bias-ply tires. This conclusion is also corroborated by the results

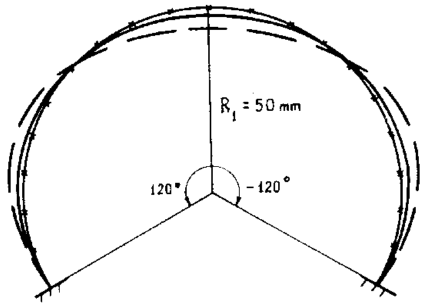

FIG. 7 — *Undeformed inner profile (—×—), nonlinear deformed (——), and linear deformed (— —) inner profiles of the tire subjected to inflation pressure.*

given in Fig. 7. Here, deformed inner profiles are obtained by using the nonlinear and linear first-order discrete-layer theory (see solid and dashed lines, respectively).

There is a great interest in the comparative analysis of the stresses and strains of an anisotropic tire obtained on the basis of the first-order Timoshenko-type model (FTM), higher-order Timoshenko-type model (HTM), first-order discrete-layer model (FDM), and higher-order discrete-layer model (HDM) already described. It should be mentioned that we have no results on the basis of the Kirchhoff-Love model. Figures 8 and 9 show the distribution of the transverse shear stresses and strains in the thickness direction at the cross section (at $\varphi = 90°$) for all aforementioned shell computational models. It can be seen that the best results are obtained by using the most general HDM model, where we must integrate the governing system of the nonlinear differential equations, the order of which is 24. It is of importance to note that due to the essentially nonuniform distribution of the transverse shear stresses σ_{13} and σ_{23} over the thickness of the tire (see Fig. 8), the FTM and HTM models do not provide the reliable prediction of tire failure. However, these simple Timoshenko-type models permit, with good accuracy, the determination of all the other components of the stress tensor.

Solving this problem typically requires 190 s, 620 s, 940 s, and 4870 s of cpu time on an IBM PC/AT 386 SX for the FTM, HTM, FDM, and HDM models, respectively.

Nonaxisymmetric Deformation of an Anisotropic Tire Subjected to Inflation Pressure and Localized Loading

Let the four-layer anisotropic toroidal shell be subjected first to uniform inflation pressure $q^{-0} = -0.15$ MPa. This problem has already been discussed. Then, the prestressed shell is subjected to normal localized loading (simulating

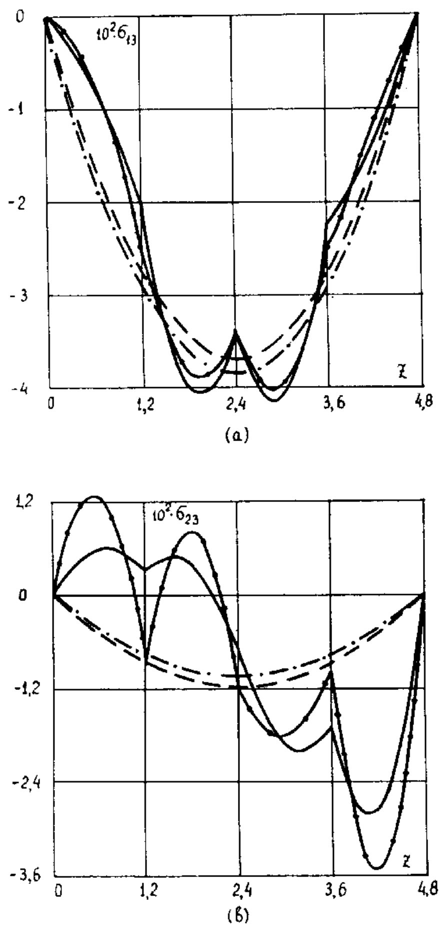

FIG. 8 — *Distribution of the stresses σ_{13} and σ_{23} in the thickness direction at the cross section (φ = 90°); FTM model (— —), HTM model (—•—), FDM model (——), and HDM model (—•—).*

the contact pressure) distributed as follows: $q^+ = -0.25$ MPa if -35 mm $\leq t \leq 35$ mm ($-0.7 \leq \varphi \leq 0.7$) and $-0.4 \leq \alpha_2 \leq 0.4$, where t is the meridional coordinate of the inner surface (Fig. 10).

Figure 11 shows the numerical results obtained by integrating the governing system in Eq 44 for nth Fourier harmonic, the order of which is 44. The solution presented here was obtained with at least 30 terms in each Fourier series and compared to results obtained using less terms. In most cases, there was a negligible difference between the 20-term solution and

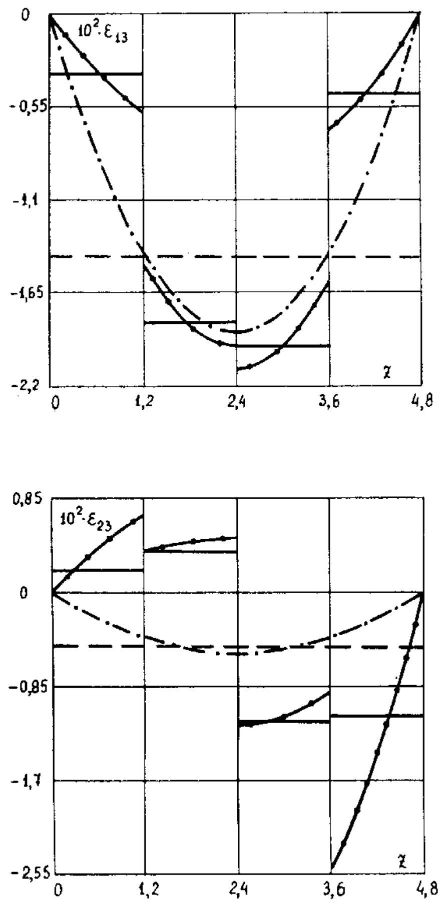

FIG. 9 — *Distribution of the strains* ϵ_{13} *and* ϵ_{23} *in the thickness direction at the cross section* (φ = 90°); *FTM model* (— —), *HTM model* (—•—), *FDM model* (——), *HDM model* (—•—).

the 30-term solution. As expected, the influence of geometrical nonlinearity is essential for small values of the circumferential coordinate α_2 (*i.e.*, for the contact zone).

Summary

Four tire computational models have been presented in this paper. One of them, the FDM, has been discussed in detail. This model is based on the independent kinematic and static hypotheses concerning the character of the

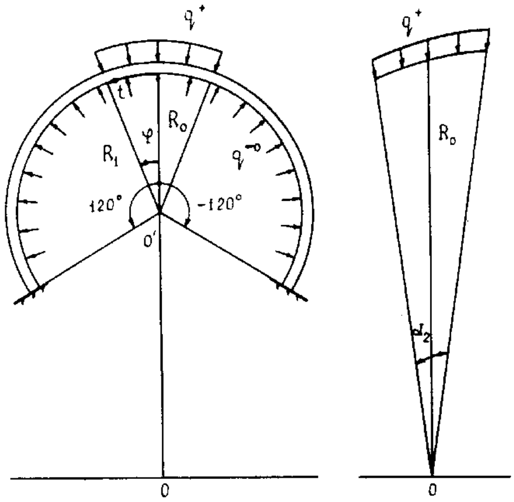

FIG. 10 — *Four-layer anisotropic tire subjected to inflation pressure and normal localized loading.*

distribution of displacements and transverse stresses over the thickness of the tire. The effects of the laminated and anisotropic material response, transverse stress and deformation nonuniformity, and geometrical nonlinearity are included. The governing equations of the theory are obtained by applying the three-field Reissner mixed variational principle. The feature of this approach is that the constitutive equations of the anisotropic elasticity theory for transverse shear stresses and strains are satisfied integrally for each layer. Using the

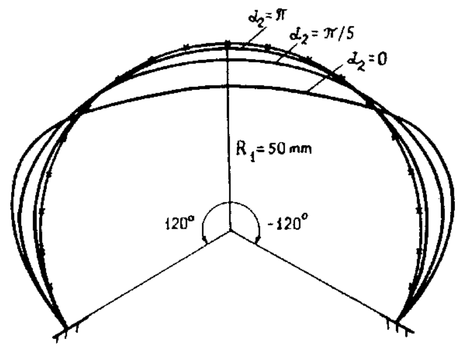

FIG. 11 — *Undeformed inner profile (—×—) and deformed inner profiles (——) of the tire subjected to inflation pressure and localized loading for various values of the circumferential coordinate* α_2.

proposed computational tire model, the joint influence of anisotropy, laminated material response, and geometrical nonlinearity on the tire stress field is investigated. Several numerical examples demonstrating various aspects of the computational model have been presented. It has been shown that neglecting the effect of anisotropy can lead to an incorrect description of stresses and strains in a four-layer tire. A comparative analysis of the transverse shear stresses and strains obtained on the basis of the four tire computational models is also given. It is important to note that none of the proposed computational models (except the HDM) requires supercomputers and that they can be run on the simplest personal computers.

Acknowledgments

This study was supported under the Tambov State Technical University Research Program. Special thanks are given to Dr. S. V. Mishchenko.

References

[*1*] Grigolyuk, E. I. and Kulikov, G. M., "Axisymmetric Deformation of Anisotropic Multilayered Shells of Revolution of Intricate Shapes," *Mechanics Composite Materials*, Vol. 17, No. 4, 1982, pp. 437–445.

[*2*] Grigolyuk, E. I. and Kulikov, G. M., *Multilayered Reinforced Shells, Analysis of Pneumatic Tires*, Moscow, 1988 (in Russian).

[*3*] Frank, F. and Hofferberth, W., "Mechanics of the Pneumatic Tire," *Rubber Chemistry and Technology*, Vol. 40, No. 1, 1967, pp. 271–322.

[*4*] Ridha, R. A., "Computation of Stresses, Strains and Deformation of Tires," *Rubber Chemistry and Technology*, Vol. 53, No. 4, 1980, pp. 849–902.

[*5*] Ridha, R. A. and Clark, S. K., "Tire Stress and Deformation," in *Mechanics of Pneumatic Tires*, Clark, S. K., ed., Washington, 1981, pp. 475–540.

[*6*] Walter, J. D., "Tire Stress and Deformation," in *Mechanics of Pneumatic Tires*, Clark, S. K., ed., Washington, 1971, pp. 406–444.

[*7*] Haas, R. and Dietzius, A., *Stoffdehnung und Formänderung der Hulle von Prall-Luftschiffen*, Untersuchungen in Luftschiffban der Siemens-Schuckert-Werke, Berlin, 1913.

[*8*] Biderman, V. L., "Computation of Inflated Profile and Stresses in the Elements of Pneumatic Tires Subjected to Internal Pressure," Moscow, Trans. Scientific Institute of Tire Industry, No. 3, 1957, pp. 16–51 (in Russian).

[*9*] Hofferberth, W., "Zur Festigkeit des Luftreifens," *Kautschuk und Gummi*, No. 9, 1956, pp. 225–231.

[*10*] Grigolyuk, E. I. and Kulikov, G. M., *Methods of Investigation of Multilayered Composite Shells with Application to Pneumatic Tires*, Scientific and Technical Progress in Machine Building, Issue 39, Moscow, 1993 (in Russian).

[*11*] Noor, A. K. and Tanner, J. A., "Advances and Trends in the Development of Computational Models for Tires," *Computers & Structures*, Vol. 20, No. 1–3, 1985, pp. 517–533.

[*12*] Böhm, F., *Zur Mechanik des Luftreifens*, Habilitationsschrift, Stuttgart, 1966.

[*13*] De Eskinazi, J., Werner, S., and Yang, T. Y., "Contact of an Inflated Toroidal Membrane with a Flat Surface as an Approach to the Tire Deflection Problem," *Tire Science and Technology*, TSTCA, Vol. 3, No. 1, 1975, pp. 43–61.

[*14*] Robecchi, E., "Mechanics of the Pneumatic Tire, The laminar Model under Inflation and in Rotation," *Tire Science and Technology*, TSTCA, Vol. 1, No. 4, 1973, pp. 382–438.

[*15*] Tielking, J. T., McIvor, I. K., and Clark, S. K., "A modified Linear Membrane Theory for the Pressurized Toroid," *J. Applied Mechanics*, Vol. 38, 1971, pp. 418–422.

[16] Böhm, F., "Mechanik des Gurtelreifens," *Ingenieur-Archiv*, Vol. 35, No. 2, 1966, pp. 82–101.

[17] Brewer, H. K., "Tire Stress and Deformation from Composite Theory," *Tire Science and Technology*, TSTCA, Vol. 1, No. 1, 1973, pp. 47–76.

[18] Biderman, V. L. and Levkovskaya, E. Ya., "Computation of Stresses in the Radial Tires Subjected to Internal Pressure," International Rubber Conference, 1969, Moscow, 1971, pp. 362–369 (in Russian).

[19] Grigolyuk, E. I. and Kulikov, G. M., "Numerical Solution of Problems Involving the Statics of Geometrically Nonlinear Anisotropic Multilayered Shells of Revolution," *Mechanics Composite Materials*, Vol. 17, No. 3, 1981, pp. 294–302.

[20] Grigolyuk, E. I., Korneichuk, L. G., Kulikov, G. M., and Nosatenko, P. Ya., "Stress-Strain State of Modern Tires," *Mekhanika Kompozitnykh Materialov*, No. 2, 1984, pp. 296–307 (in Russian).

[21] Noor, A. K., Andersen, C. M., and Tanner, J. A., "Exploiting Symmetries in the Modeling and Analysis of Tires," *Computer Methods Applied Mechanics and Engineering*, Vol. 63, No. 1, 1987, pp. 37–81.

[22] Grigolyuk, E. I. and Kulikov, G. M., "Analysis of Radial Tires on the Basis of Generalized Timoshenko-type Theory," *Mekhanika Tverdogo Tela*, No. 4, 1984, pp. 166–174 (in Russian).

[23] Grigolyuk, E. I. and Kulikov, G. M., "Effect of Shear Stress Nonuniformity in Modern Tires," *Mechanics Composite Materials*, Vol. 22, No. 5, 1987, pp. 607–613.

[24] Grigolyuk, E. I. and Kulikov, G. M., "Axisymmetric Deformation of Multilayered Anisotropic Shells of Revolution Subjected to Tangentional Loads," *Mekhanika Kompozitnykh Materialov*, No. 4, 1992, pp. 484–491 (in Russian).

[25] Kulikov, G. M., "Analysis of Pneumatic Tires on the Basis of Generalized Hypothesis of Broken Line," *Problemy Protchnosti*, No. 2, 1990, pp. 91–95 (in Russian).

[26] Kulikov, G. M., "Nonaxisymmetric Deformation of Multilayered Anisotropic Shells of Revolution Subjected to Tangential Loading," *Mekhanika Kompozitnykh Materialov*, No. 5, 1992, pp. 597–602 (in Russian).

[27] Grigolyuk, E. I. and Kulikov, G. M., "General Direction of Development of the Theory of Multilayered Shells," *Mechanics Composite Materials*, Vol. 24, No. 2, 1988, pp. 231–241.

[28] Noor, A. K. and Burton, W. S., "Assessment of Computational Models for Multilayered Composite Shells," *Applied Mechanics Reviews*, Vol. 43, No. 4, 1990, pp. 67–97.

[29] Kulikov, G. M., "Influence of Anisotropy on the Stress State of Multilayered Reinforced Shells," *Sov. Applied Mechanics*, Vol. 22, No. 12, 1987, pp. 1166–1170.

[30] Kulikov, G. M., "Thermoelasticity of Flexible Multilayered Anisotropic Shells," *Mekhanika Tverdogo Tela*, No. 2, 1994, pp. 33–42 (in Russian).