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ASSUMED STRESS-STRAIN QUADRILATERAL PLATE ELEMENTS BASED ON ANALYTICAL AND NUMERICAL INTEGRATION

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Abstract: On the basis of the first-order solid plate theory, proposed in earlier authors' articles, two assumed stress-strain ANS quadrilateral four-node solid plate elements are developed. This paper generalizes the conventional assumed stress solid plate/shell finite element formulation because transverse shear strains are distributed in the thickness direction according to the linear law. Both quadrilateral ANS solid plate elements developed are based on the unified technique that allows assessing their advantages and disadvantages. It is worth noting that element stiffness matrices of elaborated quadrilaterals including a displacement-based ANS quadrilateral four-node element have six zero eigenvalues as required for satisfaction of the general rigid-body motion representation.

1 Introduction

A robust finite element formulation pioneered by Pian [1] is the hybrid stress method. In this formulation the displacements on the element boundary are assumed to provide displacement compatibility between elements, whereas internal stresses are assumed so as to satisfy the differential equilibrium equations. Pian's work was originally based upon the principle of the stationary complementary energy. Later, an alternative assumed stress version was proposed by applying the Hellinger-Reissner variational principle that simplifies the evaluation of the element stiffness matrix [2]. In the eighties the assumed stress-strain [3] and assumed strain [4] finite element formulations were also developed. The first formulation is based on the Hu-Washizu variational principle while the second one departs from its modified version, in which only displacements and strains are chosen as fundamental unknowns.

In the last two decades, a considerable work has been carried out on three-dimensional continuum-based finite elements that can handle thin plate/shell analysis satisfactorily. These elements are typically defined by two layers of nodes at the bottom and top surfaces of the shell with three displacement degrees of freedom per node and known as solid-shell elements [5]. The development of solid-shell elements is not straightforward. In order to construct solid elements with high computational characteristics severe deficiencies such as shear, membrane, trapezoidal and thickness locking must be overcome. Currently, it is well known that for the best computational efficiency of solid quadrilaterals for the thin plate/shell analysis the assumed natural strain (**ANS**) method has to be employed. This method was originally proposed by Hughes and Tezduyar [6] for the displacement-based plate formulation and further has been used for the development of the assumed stress solid shell formulation [7, 8].

In this paper the ANS method is also applied to improve the performance of two assumed stress-strain solid plate quadrilaterals employing a unified technique. This allows one to assess their advantages and disadvantages and to compare with the corresponding displacement-based ANS solid plate element. It is remarkable that both assumed stress-strain ANS quadrilaterals exhibit an excellent performance because no expensive numerical procedures of the matrix inversion compared to assumed strain and assumed stress elements are needed. All can be carried out analytically. In order to avoid thickness locking, the simplest and robust remedy of Lee et al. [9] was used. Although a more general approach of Sze at al. [8], based on the ad hoc modified constitutive stiffness matrix, can be also applied.

2 Strain-displacement equations of first-order solid plate theory

Consider a plate of uniform thickness h. The plate may be defined as a 3D body bounded by two planes S^- and S^+, located at the distances δ^- and δ^+ measured with respect to the reference plane S^{R}, and the edge boundary cylindrical surface Ω that is perpendicular to the reference plane (Fig. 1). Let the reference plane S^{R} be referred to the Cartesian coordinate system x^1 and x^2. The x^3 – axis is oriented along the normal direction.

The position vector $\mathbf{x}$ of the arbitrarily point of the plate body can be expressed as

$$\mathbf{x} = N^-\mathbf{x}^- + N^+\mathbf{x}^+, \tag{1a}$$

$$\mathbf{x}^\pm = \mathbf{x}^{\mathrm{R}} + \delta^\pm \mathbf{e}_3, \qquad \mathbf{x}^{\mathrm{R}} = x^\alpha \mathbf{e}_\alpha, \tag{1b}$$

$$N^- = \frac{1}{h}\left(\delta^+ - x^3\right), \qquad N^+ = \frac{1}{h}\left(x^3 - \delta^-\right), \tag{1c}$$

where $\mathbf{x}^{\mathrm{R}}\left(x^1,\ x^2\right)$ is the position vector of the reference plane; $\mathbf{x}^\pm$ are the position vectors of the face planes; $N^\pm\left(x^3\right)$ are the linear through-thickness shape functions of the plate.

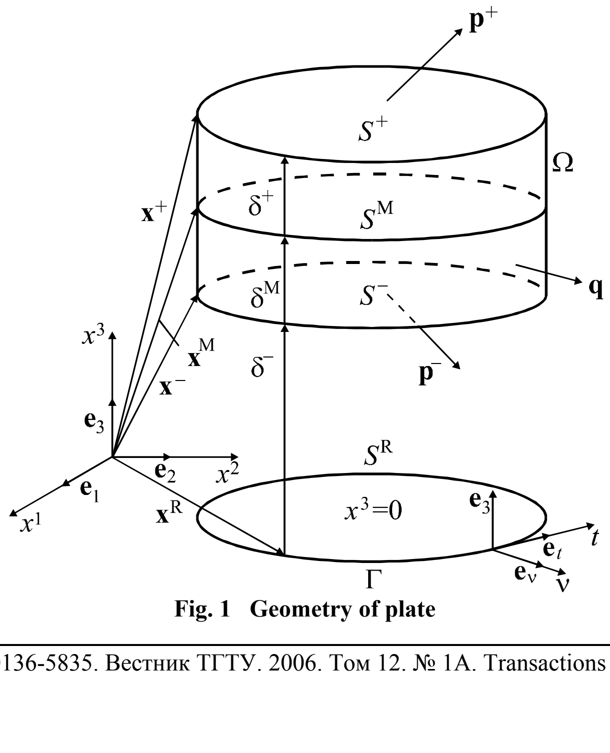


Fig. 1 Geometry of plate

The first-order solid plate theory is based on the linear approximation of displacements in the thickness direction (Timoshenko-Mindlin kinematics [10]):

$$\hat{\mathbf{x}} = N^- \hat{\mathbf{x}}^- + N^+ \hat{\mathbf{x}}^+, \tag{2a}$$

$$\hat{\mathbf{x}}^\pm = \mathbf{x}^\pm + \mathbf{u}^\pm, \tag{2b}$$

$$\mathbf{u}^\pm = u_i^\pm \mathbf{e}^i, \tag{2c}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}^\pm$ are the position vectors of points in the plate body in its current configuration; $\mathbf{u}^\pm\left(x^1,\ x^2\right)$ are the displacement vectors of the face planes; $\mathbf{e}^i = \mathbf{e}_i$ are the orthonormal base vectors in the Cartesian coordinate space.

The components of the strain tensor can be written as

$$2\varepsilon_{ij} = \hat{\mathbf{x}}_{,i} \cdot \hat{\mathbf{x}}_{,j} - \mathbf{x}_{,i} \cdot \mathbf{x}_{,j}, \tag{3}$$

where the abbreviation $(\ )_{,i}$ implies the partial derivatives with respect to coordinates x^i. Here and in the following developments indices i and j take the values 1, 2 and 3, while Greek indices α and β take the values 1 and 2.

Substituting position vectors from Eqs. (1) and (2) into 3D strain-displacement relationships (3) and retaining only geometrically linear terms, one obtains

$$\varepsilon_{\alpha\beta} = L^- \varepsilon_{\alpha\beta}^- + L^{\mathrm{M}} \varepsilon_{\alpha\beta}^{\mathrm{M}} + L^+ \varepsilon_{\alpha\beta}^+, \tag{4}$$

$$\varepsilon_{\alpha 3} = N^- \varepsilon_{\alpha 3}^- + N^+ \varepsilon_{\alpha 3}^+, \qquad \varepsilon_{33} = \overline{\varepsilon}_{33},$$

$$L^- = N^-\left(N^- - N^+\right), \qquad L^{\mathrm{M}} = 4N^- N^+, \qquad L^+ = N^+\left(N^+ - N^-\right), \tag{5}$$

where $L^\pm\left(x^3\right)$ and $L^{\mathrm{M}}\left(x^3\right)$ are the quadratic through-thickness shape functions of the plate; $\varepsilon_{\alpha\beta}^\pm$ and $\varepsilon_{\alpha\beta}^{\mathrm{M}}$ are the in-plane strains of face and middle planes, correspondingly; $\varepsilon_{\alpha 3}^\pm$ are the transverse shear strains of face planes defined as

$$2\varepsilon_{\alpha\beta}^{\mathrm{I}} = \mathbf{u}_{,\alpha}^{\mathrm{I}} \cdot \mathbf{e}_\beta + \mathbf{u}_{,\beta}^{\mathrm{I}} \cdot \mathbf{e}_\alpha = u_{\alpha,\beta}^{\mathrm{I}} + u_{\beta,\alpha}^{\mathrm{I}} \qquad (\mathrm{I} = -,\ \mathrm{M},\ +), \tag{6a}$$

$$2\varepsilon_{\alpha 3}^\pm = \boldsymbol{\beta} \cdot \mathbf{e}_\alpha + \mathbf{u}_{,\alpha}^\pm \cdot \mathbf{e}_3 = \beta_\alpha + u_{3,\alpha}^\pm, \tag{6b}$$

$$\overline{\varepsilon}_{33} = \boldsymbol{\beta} \cdot \mathbf{e}_3 = \beta_3, \tag{6c}$$

$$\boldsymbol{\beta} = \frac{1}{h}\left(\mathbf{u}^+ - \mathbf{u}^-\right), \quad \mathbf{u}^{\mathrm{M}} = \frac{1}{2}\left(\mathbf{u}^- + \mathbf{u}^+\right), \quad \boldsymbol{\beta} = \beta_i \mathbf{e}^i, \quad \mathbf{u}^{\mathrm{M}} = u_i^{\mathrm{M}} \mathbf{e}^i. \tag{6d}$$

Note that relationships (6a) and (6b) are valid because a simple formula

$$\mathbf{x}_{,\alpha}^\pm = \mathbf{e}_\alpha \tag{7}$$

holds.

The strain terms (4) quadratic in x^3 can be neglected because of their minor significance in most plate problems. So, one can write

$$\varepsilon_{\alpha\beta} = N^- \varepsilon_{\alpha\beta}^- + N^+ \varepsilon_{\alpha\beta}^+, \tag{8}$$

$$\varepsilon_{\alpha 3} = N^- \varepsilon_{\alpha 3}^- + N^+ \varepsilon_{\alpha 3}^+, \qquad \varepsilon_{33} = \overline{\varepsilon}_{33}.$$

Strain-displacement equations (8) were used extensively for the development of robust solid plate/shell elements [11-13]. A close finite element formulation, in which in-plane and transverse shear strains depend linearly on the thickness coordinate, was proposed by Lee et al. [9].

It is important that transverse components of the strain tensor (6b) and (6c) satisfy the following coupling conditions:

$$2\left(\varepsilon_{\alpha 3}^{+}-\varepsilon_{\alpha 3}^{-}\right)=h\bar{\varepsilon}_{33,\alpha}. \qquad (9)$$

A proof of this statement is obvious and was given for the finite deformation plate theory by Kulikov and Plotnikova [13].

3 Displacement-based ANS quadrilateral solid plate element

For the isoparametric quadrilateral four-node solid plate element the position vector in the initial configuration and the displacement vector are approximated according to the standard C$^{\text{o}}$ interpolation

$$\mathbf{x}^{\mathrm{R}}=\sum_r N_r \mathbf{x}_r^{\mathrm{R}}, \qquad (10)$$

$$\mathbf{x}^{\mathrm{R}}=\left[x^1\ x^2\ 0\right]^{\mathrm{T}}, \qquad \mathbf{x}_r^{\mathrm{R}}=\left[x_r^1\ x_r^2\ 0\right]^{\mathrm{T}} \qquad (11)$$

and

$$\mathbf{u}=\sum_r N_r \mathbf{u}_r, \qquad (12)$$

$$\mathbf{u}=\left[u_1^-\ u_1^+\ u_2^-\ u_2^+\ u_3^-\ u_3^+\right]^{\mathrm{T}}, \quad \mathbf{u}_r=\left[u_{1r}^-\ u_{1r}^+\ u_{2r}^-\ u_{2r}^+\ u_{3r}^-\ u_{3r}^+\right]^{\mathrm{T}}, \qquad (13)$$

where $\mathbf{u}_r$ are the displacement vectors of the element nodes (Fig. 2); $N_r\left(\xi^1, \xi^2\right)$ are the bilinear shape functions of the element; ξ^α are the natural coordinates; the index r runs from 1 to 4 and denotes a number of nodes.

In order to avoid shear locking of the displacement-based formulation, we employ the ANS method [6] using its non-conventional treatment (compared to [7, 8]) because herein transverse shear strains of face planes (6b) have to be interpolated

$$\varepsilon_{\alpha 3}^{\pm}=\ell_\alpha^\beta \hat{\varepsilon}_{\beta 3}^{\pm}, \qquad (14)$$

$$\hat{\varepsilon}_{13}^{\pm}=\frac{1}{2}\left(1-\xi^2\right)\hat{\varepsilon}_{13}^{\pm}(\mathrm{B})+\frac{1}{2}\left(1+\xi^2\right)\hat{\varepsilon}_{13}^{\pm}(\mathrm{D}), \qquad (15)$$

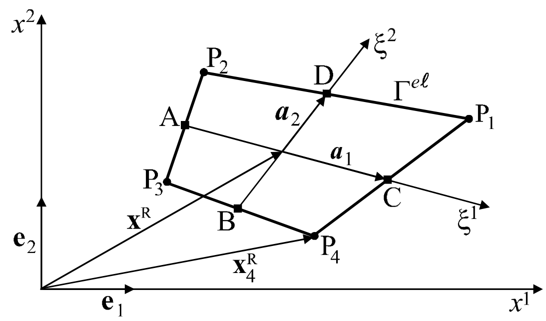


Fig. 2 Quadrilateral plate element

$$\hat{\varepsilon}^{\pm}_{23} = \frac{1}{2}\left(1-\xi^1\right)\hat{\varepsilon}^{\pm}_{23}(\mathrm{A}) + \frac{1}{2}\left(1+\xi^1\right)\hat{\varepsilon}^{\pm}_{23}(\mathrm{C}) \ ,$$

where $\hat{\varepsilon}^{\pm}_{\alpha 3}$ are the covariant transverse shear components of the strain tensor in the contravariant basis $\boldsymbol{a}^{\alpha}$, $\mathbf{e}^3$ given by a standard relation $\boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}^{\beta} = \delta^{\beta}_{\alpha}$, that is,

$$\boldsymbol{a}_{\alpha} = t^{\beta}_{\alpha}\mathbf{e}_{\beta}, \quad \boldsymbol{a}^{\alpha} = \ell^{\alpha}_{\beta}\mathbf{e}^{\beta}, \tag{16}$$

$$t^{\alpha}_1 = \frac{1}{4}\left(1+\xi^2\right)\left(x^{\alpha}_1 - x^{\alpha}_2\right) + \frac{1}{4}\left(1-\xi^2\right)\left(x^{\alpha}_4 - x^{\alpha}_3\right) \ , \tag{17}$$

$$t^{\alpha}_2 = \frac{1}{4}\left(1+\xi^1\right)\left(x^{\alpha}_1 - x^{\alpha}_4\right) + \frac{1}{4}\left(1-\xi^1\right)\left(x^{\alpha}_2 - x^{\alpha}_3\right) \ ,$$

$$\ell^1_1 = \frac{1}{\Lambda}t^2_2, \quad \ell^2_1 = -\frac{1}{\Lambda}t^2_1, \quad \ell^1_2 = -\frac{1}{\Lambda}t^1_2, \quad \ell^2_2 = \frac{1}{\Lambda}t^1_1.$$

Here, $\Lambda = \det \mathbf{J}$ denotes the Jacobian, i.e., a determinant of the transformation matrix $\mathbf{J} = \left[t^{\beta}_{\alpha}\right]$. These strains are evaluated at the sampling points A, B, C and D, located at the center of each edge, as

$$2\hat{\varepsilon}^{\pm}_{13}(\mathrm{B}) = \frac{1}{4h}\left(x^{\alpha}_4 - x^{\alpha}_3\right)\left(u^{+}_{\alpha 3} - u^{-}_{\alpha 3} + u^{+}_{\alpha 4} - u^{-}_{\alpha 4}\right) + \frac{1}{2}\left(u^{\pm}_{34} - u^{\pm}_{33}\right) \ , \tag{18}$$

$$2\hat{\varepsilon}^{\pm}_{13}(\mathrm{D}) = \frac{1}{4h}\left(x^{\alpha}_1 - x^{\alpha}_2\right)\left(u^{+}_{\alpha 1} - u^{-}_{\alpha 1} + u^{+}_{\alpha 2} - u^{-}_{\alpha 2}\right) + \frac{1}{2}\left(u^{\pm}_{31} - u^{\pm}_{32}\right) \ ,$$

$$2\hat{\varepsilon}^{\pm}_{23}(\mathrm{A}) = \frac{1}{4h}\left(x^{\alpha}_2 - x^{\alpha}_3\right)\left(u^{+}_{\alpha 2} - u^{-}_{\alpha 2} + u^{+}_{\alpha 3} - u^{-}_{\alpha 3}\right) + \frac{1}{2}\left(u^{\pm}_{32} - u^{\pm}_{33}\right) \ ,$$

$$2\hat{\varepsilon}^{\pm}_{23}(\mathrm{C}) = \frac{1}{4h}\left(x^{\alpha}_1 - x^{\alpha}_4\right)\left(u^{+}_{\alpha 1} - u^{-}_{\alpha 1} + u^{+}_{\alpha 4} - u^{-}_{\alpha 4}\right) + \frac{1}{2}\left(u^{\pm}_{31} - u^{\pm}_{34}\right) \ .$$

Substituting displacement interpolations (12) in strain-displacement relationships (6a) and (6c), and taking into account Eqs. (14), (15) and (18), one obtains

$$\boldsymbol{\mathcal{E}} = \mathbf{BU}, \tag{19a}$$

$$\mathbf{B} = \left[\mathbf{B}_1 \ \mathbf{B}_2 \ \mathbf{B}_3 \ \mathbf{B}_4\right], \quad \mathbf{U} = \left[\mathbf{u}^{\mathrm{T}}_1 \ \mathbf{u}^{\mathrm{T}}_2 \ \mathbf{u}^{\mathrm{T}}_3 \ \mathbf{u}^{\mathrm{T}}_4\right]^{\mathrm{T}}, \tag{19b}$$

$$\boldsymbol{\mathcal{E}} = \left[\varepsilon^{-}_{11} \ \varepsilon^{+}_{11} \ \varepsilon^{-}_{22} \ \varepsilon^{+}_{22} \ 2\varepsilon^{-}_{12} \ 2\varepsilon^{+}_{12} \ 2\varepsilon^{-}_{13} \ 2\varepsilon^{+}_{13} \ 2\varepsilon^{-}_{23} \ 2\varepsilon^{+}_{23} \ \bar{\varepsilon}_{33}\right]^{\mathrm{T}}.$$

As usual in the finite element literature $\mathbf{B_r}$ denote the matrices together constituting the strain-displacement transformation matrix $\mathbf{B}$ of order 11×24, which may be evaluated in the standard manner.

A variational equation for the displacement-based finite element formulation can be written as

$$\int_{-1}^{1}\int_{-1}^{1}\left(\delta\boldsymbol{\mathcal{E}}^{\mathrm{T}}\mathbf{D}\boldsymbol{\mathcal{E}} - \delta\mathbf{u}^{\mathrm{T}}\boldsymbol{\mathcal{P}}\right)\Lambda d\xi^1 d\xi^2 - \oint_{\Gamma^{e\ell}} \delta\mathbf{u}^{\mathrm{T}}_{\Gamma}\,\overset{\circ}{\mathbf{H}}_{\Gamma}\, ds = 0, \tag{20a}$$

$$\boldsymbol{\mathcal{P}} = \left[-p^{-}_1 \ p^{+}_1 - p^{-}_2 \ p^{+}_2 - p^{-}_3 \ p^{+}_3\right]^{\mathrm{T}}, \quad \mathbf{u}_{\Gamma} = \left[u^{-}_{\nu} \ u^{+}_{\nu} \ u^{-}_{t} \ u^{+}_{t} \ u^{-}_{3} \ u^{+}_{3}\right]^{\mathrm{T}}, \tag{20b}$$

$$\overset{\circ}{\mathbf{H}}_{\Gamma} = \left[\overset{\circ}{H}{}^{-}_{\nu\nu} \; \overset{\circ}{H}{}^{+}_{\nu\nu} \; \overset{\circ}{H}{}^{-}_{\nu t} \; \overset{\circ}{H}{}^{+}_{\nu t} \; \overset{\circ}{H}{}^{-}_{\nu 3} \; \overset{\circ}{H}{}^{+}_{\nu 3} \right]^{\mathrm{T}},$$

where **D** is the constitutive stiffness matrix of order 11×11 [14], whose components are found in accordance with the simplest remedy of Lee et al. [9] to prevent thickness locking; $u^{\pm}_{\nu}$, $u^{\pm}_{t}$ and $u^{\pm}_{3}$ are the components of displacement vectors of the face planes in the orthonormal basis $\mathbf{e}_{\nu}$, $\mathbf{e}_t$ and $\mathbf{e}_3$, associated with the bounding curve $\Gamma^{e\ell}$ (Fig. 2); $\overset{\circ}{H}{}^{\pm}_{\nu\nu}$, $\overset{\circ}{H}{}^{\pm}_{\nu t}$ and $\overset{\circ}{H}{}^{\pm}_{\nu 3}$ are the components of the load resultant tensors in the same orthonormal basis defined as

$$\overset{\circ}{H}{}^{\pm}_{\nu æ} = \int_{\delta^-}^{\delta^+} q_{æ} N^{\pm} dx^3 \quad (æ = \nu,\ t \text{ and } 3) \ . \tag{21}$$

Using approximations (12) and (19) into the variational equation (20) yields the element equilibrium equations

$$\mathbf{K}_{\mathrm{D}}\mathbf{U} = \mathbf{F}, \tag{22}$$

where **F** is the force vector and $\mathbf{K}_{\mathrm{D}}$ is the elemental stiffness matrix of order 24×24:

$$\mathbf{K}_{\mathrm{D}} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\Lambda d\xi^1 d\xi^2. \tag{23}$$

Here and in the following studies the integrals are calculated by employing a Gauss integration scheme with 2×2 integration points except for a case of the analytical evaluation of matrices **Q** and $\tilde{\mathbf{Q}}$ in the next section.

4 Assumed stress-strain ANS quadrilateral solid plate element

The assumed stress-strain finite element formulation developed is based on the fundamental approximations of displacements (2) and displacement-dependent strains (8) in the thickness direction. Additionally, one has to adopt the similar approximation for the assumed displacement-independent strains:

$$\varepsilon^{\mathrm{AS}}_{\alpha\beta} = N^- E^-_{\alpha\beta} + N^+ E^+_{\alpha\beta}, \tag{24}$$

$$\varepsilon^{\mathrm{AS}}_{\alpha 3} = N^- E^-_{\alpha 3} + N^+ E^+_{\alpha 3}, \qquad \varepsilon^{\mathrm{AS}}_{33} = E_{33}.$$

Substituting approximations (2), (8) and (24) into the 3D Hu-Washizu variational principle [13] and introducing stress resultants

$$H^{\pm}_{\alpha\beta} = \int_{\delta^-}^{\delta^+} \sigma_{\alpha\beta} N^{\pm} dx^3, \quad H^{\pm}_{\alpha 3} = \int_{\delta^-}^{\delta^+} \sigma_{\alpha 3} N^{\pm} dx^3, \quad H_{33} = \int_{\delta^-}^{\delta^+} \sigma_{33} dx^3, \tag{25}$$

one derives

$$\int_{-1}^{1}\int_{-1}^{1} \left[\delta\mathbf{E}^{\mathrm{T}}(\mathbf{H} - \mathbf{D}\mathbf{E}) + \delta\mathbf{H}^{\mathrm{T}}(\mathbf{E} - \boldsymbol{\mathcal{E}}) - \delta\boldsymbol{\mathcal{E}}^{\mathrm{T}}\mathbf{H} + \delta\mathbf{u}^{\mathrm{T}}\boldsymbol{\mathcal{P}} \right] \Lambda \mathrm{d}\xi^1 \mathrm{d}\xi^2 \tag{26a}$$

$$+ \oint_{\Gamma^{\mathrm{e}\ell}} \delta\mathbf{u}^{\mathrm{T}}_{\Gamma} \overset{\circ}{\mathbf{H}}_{\Gamma}\, \mathrm{ds} = 0,$$

$$\mathbf{E} = \left[E_{11}^{-} \; E_{11}^{+} \; E_{22}^{-} \; E_{22}^{+} \; 2E_{12}^{-} \; 2E_{12}^{+} \; 2E_{13}^{-} \; 2E_{13}^{+} \; 2E_{23}^{-} \; 2E_{23}^{+} \; E_{33} \right]^{\mathrm{T}}, \tag{26b}$$

$$\mathbf{H} = \left[H_{11}^{-} \; H_{11}^{+} \; H_{22}^{-} \; H_{22}^{+} \; H_{12}^{-} \; H_{12}^{+} \; H_{13}^{-} \; H_{13}^{+} \; H_{23}^{-} \; H_{23}^{+} \; H_{33} \right]^{\mathrm{T}}.$$

The remaining matrix notations are presented by Eqs. (13), (19b) and (20b).

4.1 Consistent assumed stress-strain formulation

In order to fulfill a patch test, the assumed stress resultants are interpolated by invoking ideas of Pian and Sumihara [15] and Simo et al. [7]

$$\mathbf{H} = \mathbf{P}_{\mathrm{H}} \mathbf{\Psi}, \quad \mathbf{\Psi} = \left[\psi_s \; \psi_{11+s} \right]^{\mathrm{T}}, \tag{27a}$$

$$\mathbf{P}_{\mathrm{H}} = \left[\mathbf{I}_{11\times 11} \; \mathbf{P}_{\mathrm{H}}^{\mathrm{inp}} \; \mathbf{P}_{\mathrm{H}}^{\mathrm{trs}} \; \mathbf{P}_{\mathrm{H}}^{\mathrm{trn}} \right], \quad \mathbf{P}_{\mathrm{H}}^{\mathrm{trn}} = \begin{bmatrix} \mathbf{O}_{10\times 1} & \mathbf{O}_{10\times 1} & \mathbf{O}_{10\times 1} \\ \bar{\xi}^1 & \bar{\xi}^2 & \bar{\xi}^1 \bar{\xi}^2 \end{bmatrix}, \tag{27b}$$

$$\mathbf{P}_{\mathrm{H}}^{\mathrm{inp}} = \begin{bmatrix} \bar{t}_1^1 \bar{t}_1^1 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{t}_2^1 \bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^1 \bar{t}_1^1 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{t}_2^1 \bar{\xi}^1 \\ \bar{t}_1^2 \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^2 \bar{t}_2^2 \bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^2 \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^2 \bar{t}_2^2 \bar{\xi}^1 \\ \bar{t}_1^1 \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{t}_2^2 \bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^1 \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{t}_2^2 \bar{\xi}^1 \\ \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} \end{bmatrix}, \quad \mathbf{P}_{\mathrm{H}}^{\mathrm{trs}} = \begin{bmatrix} \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} \\ \bar{t}_1^1 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^1 \bar{\xi}^2 & 0 & \bar{t}_2^1 \bar{\xi}^1 \\ \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^2 \bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^2 \bar{\xi}^2 & 0 & \bar{t}_2^2 \bar{\xi}^1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\xi}^\alpha = \xi^\alpha - \xi_c^\alpha, \quad \xi_c^\alpha = \frac{1}{4a} \int_{-1}^{1} \int_{-1}^{1} \xi^\alpha \Lambda d\xi^1 d\xi^2 = \frac{1}{3} \frac{b_\alpha}{a},$$

where $\mathbf{I}_{11\times 11}$ is the identity matrix; $\mathbf{O}_{5\times 1}$ and $\mathbf{O}_{10\times 1}$ are the zero vectors; the index s runs from 1 to 11. Therefore, a vector of unknown stress parameters $\mathbf{\Psi}$ contains 11 parameters for describing homogeneous states of stress resultants and 11 higher approximation modes. The transformation coefficients $\bar{t}_\alpha^\beta$ in (27b) denote the components of the Jacobian matrix $\mathbf{J}$ evaluated at the element center, whereas the symbol a denotes its determinant Λ calculated at the same point given by

$$\bar{t}_1^\alpha = \frac{1}{4}\left(x_1^\alpha - x_2^\alpha - x_3^\alpha + x_4^\alpha \right), \quad \bar{t}_2^\alpha = \frac{1}{4}\left(x_1^\alpha + x_2^\alpha - x_3^\alpha - x_4^\alpha \right), \tag{28}$$

$$\Lambda = a + b_\alpha \xi^\alpha, \quad a = \frac{1}{8}\left[\left(x_1^1 - x_3^1 \right)\left(x_2^2 - x_4^2 \right) - \left(x_2^1 - x_4^1 \right)\left(x_1^2 - x_3^2 \right) \right], \tag{29}$$

$$b_1 = \frac{1}{8}\left[\left(x_1^1 - x_2^1 \right)\left(x_3^2 - x_4^2 \right) - \left(x_3^1 - x_4^1 \right)\left(x_1^2 - x_2^2 \right) \right],$$

$$b_2 = \frac{1}{8}\left[\left(x_1^1 - x_4^1 \right)\left(x_2^2 - x_3^2 \right) - \left(x_2^1 - x_3^1 \right)\left(x_1^2 - x_4^2 \right) \right].$$

The purpose of introducing parameters $\bar{\xi}^\alpha$ lies in the simplicity of the fundamental matrix of the assumed stress-strain method, denoted below by **Q**, because a useful formula

$$\int_{-1}^{1}\int_{-1}^{1} \bar{\xi}^\alpha \Lambda d\xi^1 d\xi^2 = 0 \tag{30}$$

holds. It should be remarked that approximations (27) correspond to a transformation of the contravariant components of the stress resultant tensor in the covariant basis $\boldsymbol{a}_\alpha$, $\mathbf{e}_3$ to the Cartesian components in the basis $\mathbf{e}_\alpha$, $\mathbf{e}_3$.

The assumed displacement-independent strains are interpolated by a similar way

$$\mathbf{E} = \mathbf{P}_\mathrm{E}\boldsymbol{\Phi}, \quad \boldsymbol{\Phi} = \left[\varphi_s \ \varphi_{11+s}\right]^\mathrm{T}, \tag{31a}$$

$$\mathbf{P}_\mathrm{E} = \left[\mathbf{I}_{11\times 11}\ \mathbf{P}_\mathrm{E}^\mathrm{inp}\ \mathbf{P}_\mathrm{E}^\mathrm{trs}\ \mathbf{P}_\mathrm{E}^\mathrm{trn}\right], \quad \mathbf{P}_\mathrm{E}^\mathrm{trn} = \mathbf{P}_\mathrm{H}^\mathrm{trn}, \tag{31b}$$

$$\mathbf{P}_\mathrm{E}^\mathrm{inp} = \begin{bmatrix} \bar{\ell}_1^1\bar{\ell}_1^1\bar{\xi}^2 & 0 & \bar{\ell}_1^2\bar{\ell}_1^2\bar{\xi}^1 & 0 \\ 0 & \bar{\ell}_1^1\bar{\ell}_1^1\bar{\xi}^2 & 0 & \bar{\ell}_1^2\bar{\ell}_1^2\bar{\xi}^1 \\ \bar{\ell}_2^1\bar{\ell}_2^1\bar{\xi}^2 & 0 & \bar{\ell}_2^2\bar{\ell}_2^2\bar{\xi}^1 & 0 \\ 0 & \bar{\ell}_2^1\bar{\ell}_2^1\bar{\xi}^2 & 0 & \bar{\ell}_2^2\bar{\ell}_2^2\bar{\xi}^1 \\ 2\bar{\ell}_1^1\bar{\ell}_2^1\bar{\xi}^2 & 0 & 2\bar{\ell}_1^2\bar{\ell}_2^2\bar{\xi}^1 & 0 \\ 0 & 2\bar{\ell}_1^1\bar{\ell}_2^1\bar{\xi}^2 & 0 & 2\bar{\ell}_1^2\bar{\ell}_2^2\bar{\xi}^1 \\ \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} & \mathbf{O}_{5\times 1} \end{bmatrix}, \quad \mathbf{P}_\mathrm{E}^\mathrm{trs} = \begin{bmatrix} \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} & \mathbf{O}_{6\times 1} \\ \bar{\ell}_1^1\bar{\xi}^2 & 0 & \bar{\ell}_1^2\bar{\xi}^1 & 0 \\ 0 & \bar{\ell}_1^1\bar{\xi}^2 & 0 & \bar{\ell}_1^2\bar{\xi}^1 \\ \bar{\ell}_2^1\bar{\xi}^2 & 0 & \bar{\ell}_2^2\bar{\xi}^1 & 0 \\ 0 & \bar{\ell}_2^1\bar{\xi}^2 & 0 & \bar{\ell}_2^2\bar{\xi}^1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\boldsymbol{\Phi}$ denotes the vector of unknown strain parameters of order 22×1, whereas transformation coefficients $\bar{\ell}_\alpha^\beta$ denote the components of the Jacobian matrix $\mathbf{J}^{-1}$ evaluated also at the element center defined as

$$\bar{\ell}_1^1 = \frac{1}{a}\bar{t}_2^2, \quad \bar{\ell}_1^2 = -\frac{1}{a}\bar{t}_1^2, \quad \bar{\ell}_2^1 = -\frac{1}{a}\bar{t}_2^1, \quad \bar{\ell}_2^2 = \frac{1}{a}\bar{t}_1^1. \tag{32}$$

The approximations (31) correspond to a transformation of the covariant components of the displacement-independent strain tensor in the contravariant basis $\boldsymbol{a}^\alpha$, $\mathbf{e}^3$ to its components in the orthonormal basis $\mathbf{e}_\alpha$, $\mathbf{e}_3$.

Inserting interpolations (12), (19), (27) and (31) into the mixed variational equation (26) and introducing matrix notations

$$\mathbf{Q} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_\mathrm{H}^\mathrm{T}\mathbf{P}_\mathrm{E}\Lambda d\xi^1 d\xi^2, \tag{33a}$$

$$\mathbf{S}_\mathrm{E} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_\mathrm{E}^\mathrm{T}\mathbf{D}\mathbf{P}_\mathrm{E}\Lambda d\xi^1 d\xi^2, \quad \mathbf{R}_\mathrm{H} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_\mathrm{H}^\mathrm{T}\mathbf{B}\Lambda d\xi^1 d\xi^2, \tag{33b}$$

the following element equilibrium equations are obtained:

$$\mathbf{Q}^\mathrm{T}\boldsymbol{\Psi} = \mathbf{S}_\mathrm{E}\boldsymbol{\Phi}, \quad \mathbf{Q}\boldsymbol{\Phi} = \mathbf{R}_\mathrm{H}\mathbf{U}, \quad \mathbf{R}_\mathrm{H}^\mathrm{T}\boldsymbol{\Psi} = \mathbf{F}. \tag{34}$$

Eliminating assumed stress and strain parameter vectors $\mathbf{\Psi}$ and $\mathbf{\Phi}$ from Eq. (34), one finds

$$\mathbf{K}_{\mathrm{EH}}\mathbf{U} = \mathbf{F}. \tag{35}$$

Here, $\mathbf{K}_{\mathrm{EH}}$ denotes the element stiffness matrix given by

$$\mathbf{K}_{\mathrm{EH}} = \mathbf{R}_{\mathrm{H}}^{\mathrm{T}}\mathbf{Q}^{-\mathrm{T}}\mathbf{S}_{\mathrm{E}}\mathbf{Q}^{-1}\mathbf{R}_{\mathrm{H}}. \tag{36}$$

The fundamental matrix $\mathbf{Q}$ of the assumed stress-strain method can be found employing the analytical integration. Using Eqs. (27b), (29) and (31b) into (33a) and integrating, we obtain

$$\mathbf{Q} = \frac{4}{9}\begin{bmatrix} 9a\mathbf{I}_{11\times11} & \mathbf{O}_{11\times4} & \mathbf{O}_{11\times4} & \mathbf{O}_{11\times3} \\ \mathbf{O}_{4\times11} & \mathbf{Q}_{4\times4}^{\mathrm{inp}} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times3} \\ \mathbf{O}_{4\times11} & \mathbf{O}_{4\times4} & \mathbf{Q}_{4\times4}^{\mathrm{trs}} & \mathbf{O}_{4\times3} \\ \mathbf{O}_{3\times11} & \mathbf{O}_{3\times4} & \mathbf{O}_{3\times4} & \mathbf{Q}_{3\times3}^{\mathrm{trn}} \end{bmatrix}, \tag{37}$$

$$\mathbf{Q}_{4\times4}^{\mathrm{inp}} = \mathbf{Q}_{4\times4}^{\mathrm{trs}} = \begin{bmatrix} d_{22} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{11} & 0 \\ 0 & 0 & 0 & d_{11} \end{bmatrix}, \quad \mathbf{Q}_{3\times3}^{\mathrm{trn}} = \begin{bmatrix} d_{11} & d_{12} & b_2 \\ d_{12} & d_{22} & b_1 \\ b_2 & b_1 & a \end{bmatrix},$$

where

$$d_{\alpha\beta} = 3a\delta_{\alpha\beta} - \frac{1}{a}b_\alpha b_\beta.$$

It is seen that the matrix $\mathbf{Q}$ has a diagonal structure except for the submatrix $\mathbf{Q}_{33}^{\mathrm{trn}}$. Therefore, its inversion may be done readily, since only a submatrix $\mathbf{Q}_{33}^{\mathrm{trn}}$ has to be inverted.

Using a link between transverse components of the displacement-dependent strain tensor (9) and following a technique developed in paper [16], one can derive *four* coupling conditions for the transverse components of the displacement-independent strain tensor. These conditions imply that only 18 assumed strain parameters are *independent* of 22 ones introduced in the approximation (31). In a result, the elemental stiffness matrix has six, and only six, zero eigenvalues as required for satisfaction of the general rigid-body motion representation, since 24 displacement degrees of freedom are introduced.

As it pointed out previously, the evaluation of all remaining matrices including $\mathbf{S}_{\mathrm{E}}$ and $\mathbf{R}_{\mathrm{H}}$ are carried out numerically employing a Gauss integration scheme. But all can be done analytically due to a property of the strain-displacement transformation matrix, that is,

$$\mathbf{B} = \frac{1}{\Lambda}\tilde{\mathbf{B}}, \tag{38}$$

where $\tilde{\mathbf{B}}$ denotes a new strain-displacement transformation matrix, whose components already depend only on the polynomials of coordinates ξ^1 and ξ^2. The numerical examples showed that CPU time required for the formation of the global stiffness matrix is slightly depend on a number of elements used for discretizing a plate and the numerically integrated element matrix needs about 50 % more time than the analytically integrated one.

Unfortunately, a consistent assumed stress-strain element developed is a bit stiff when we use coarse skewed meshes in some discriminating benchmark problems. The better performance one can achieve by employing an enhanced assumed stress-strain element. We suppose that this unexpected effect is a result of applying mixed approximations of two different types (27) and (31) containing simultaneously both transformation coefficients $\bar{t}_\alpha^\beta$ and $\bar{\ell}_\alpha^\beta$.

4.2 Enhanced assumed stress-strain formulation

The patch test can be also fulfilled by employing for the displacement-dependent strains instead of a consistently assumed approximation (31) the more robust approximation given by

$$\mathbf{E} = \tilde{\mathbf{P}}_\mathrm{E}\mathbf{\Phi}, \tag{39a}$$

$$\tilde{\mathbf{P}}_\mathrm{E} = \left[\mathbf{I}_{11\times11}\ \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{inp}\ \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{trs}\ \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{trn}\right]^\mathrm{T}, \quad \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{trs} = \mathbf{P}_\mathrm{H}^\mathrm{trs}, \quad \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{trn} = \mathbf{P}_\mathrm{H}^\mathrm{trn}, \tag{39b}$$

$$\tilde{\mathbf{P}}_\mathrm{E}^\mathrm{inp} = \begin{bmatrix} \bar{t}_1^1\bar{t}_1^1\bar{\xi}^2 & 0 & \bar{t}_2^1\bar{t}_2^1\bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^1\bar{t}_1^1\bar{\xi}^2 & 0 & \bar{t}_2^1\bar{t}_2^1\bar{\xi}^1 \\ \bar{t}_1^2\bar{t}_1^2\bar{\xi}^2 & 0 & \bar{t}_2^2\bar{t}_2^2\bar{\xi}^1 & 0 \\ 0 & \bar{t}_1^2\bar{t}_1^2\bar{\xi}^2 & 0 & \bar{t}_2^2\bar{t}_2^2\bar{\xi}^1 \\ 2\bar{t}_1^1\bar{t}_1^2\bar{\xi}^2 & 0 & 2\bar{t}_2^1\bar{t}_2^2\bar{\xi}^1 & 0 \\ 0 & 2\bar{t}_1^1\bar{t}_1^2\bar{\xi}^2 & 0 & 2\bar{t}_2^1\bar{t}_2^2\bar{\xi}^1 \\ \mathbf{O}_{5\times1} & \mathbf{O}_{5\times1} & \mathbf{O}_{5\times1} & \mathbf{O}_{5\times1} \end{bmatrix}.$$

The motivation for such kind of approximations lies in the possibility to use only transformation coefficients $\bar{t}_\alpha^\beta$, which correspond to a transformation of the contravariant tensor components to the Cartesian ones.

Following a technique developed in the previous subsection, one derives a system of finite element equations

$$\tilde{\mathbf{K}}_\mathrm{EH}\mathbf{U} = \mathbf{F}, \tag{40}$$

where

$$\tilde{\mathbf{K}}_\mathrm{EH} = \mathbf{R}_\mathrm{H}^\mathrm{T}\tilde{\mathbf{Q}}^{-\mathrm{T}}\tilde{\mathbf{S}}_\mathrm{E}\tilde{\mathbf{Q}}^{-1}\mathbf{R}_\mathrm{H}, \tag{41}$$

$$\tilde{\mathbf{Q}} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_\mathrm{H}^\mathrm{T}\tilde{\mathbf{P}}_\mathrm{E}\Lambda d\xi^1 d\xi^2, \qquad \tilde{\mathbf{S}}_\mathrm{E} = \int_{-1}^{1}\int_{-1}^{1} \tilde{\mathbf{P}}_\mathrm{E}^\mathrm{T}\mathbf{D}\tilde{\mathbf{P}}_\mathrm{E}\Lambda d\xi^1 d\xi^2. \tag{42}$$

The fundamental matrix $\tilde{\mathbf{Q}}$ can be found again in a closed form by using the analytical integration. Taking into account Eqs. (27b), (29), (39b) and (42), we obtain slightly more complex formula:

$$\tilde{\mathbf{Q}} = \frac{4}{9}\begin{bmatrix} 9a\mathbf{I}_{11\times11} & \mathbf{O}_{11\times4} & \mathbf{O}_{11\times4} & \mathbf{O}_{11\times3} \\ \mathbf{O}_{4\times11} & \tilde{\mathbf{Q}}_{4\times4}^\mathrm{inp} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times3} \\ \mathbf{O}_{4\times11} & \mathbf{O}_{4\times4} & \tilde{\mathbf{Q}}_{4\times4}^\mathrm{trs} & \mathbf{O}_{4\times3} \\ \mathbf{O}_{3\times11} & \mathbf{O}_{3\times4} & \mathbf{O}_{3\times4} & \tilde{\mathbf{Q}}_{3\times3}^\mathrm{trn} \end{bmatrix}, \tag{43}$$

$$\tilde{\mathbf{Q}}^{\text{inp}}_{4\times4} = \begin{bmatrix} \pi_{11}^2 d_{22} & 0 & \pi_{12}^2 d_{12} & 0 \\ 0 & \pi_{11}^2 d_{22} & 0 & \pi_{12}^2 d_{12} \\ \pi_{12}^2 d_{12} & 0 & \pi_{22}^2 d_{11} & 0 \\ 0 & \pi_{12}^2 d_{12} & 0 & \pi_{22}^2 d_{11} \end{bmatrix},$$

$$\tilde{\mathbf{Q}}^{\text{trs}}_{4\times4} = \begin{bmatrix} \pi_{11} d_{22} & 0 & \pi_{12} d_{12} & 0 \\ 0 & \pi_{11} d_{22} & 0 & \pi_{12} d_{12} \\ \pi_{12} d_{12} & 0 & \pi_{22} d_{11} & 0 \\ 0 & \pi_{12} d_{12} & 0 & \pi_{22} d_{11} \end{bmatrix}, \quad \tilde{\mathbf{Q}}^{\text{trn}}_{3\times3} = \mathbf{Q}^{\text{trn}}_{3\times3},$$

where

$$\pi_{\alpha\beta} = \bar{t}^1_\alpha \bar{t}^1_\beta + \bar{t}^2_\alpha \bar{t}^2_\beta .$$

5 Numerical examples

The performance of the proposed quadrilateral four-node ANS solid plate elements is evaluated with several problems extracted from the literature. A listing of these elements and the abbreviations used to identify them are contained in Table 1.

Table 1

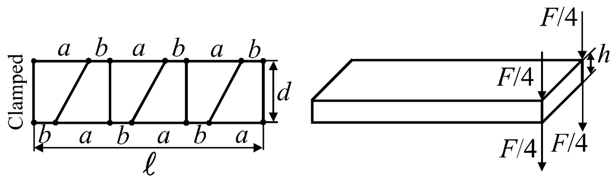
Listing of quadrilateral four-node solid plate elements

Name	Description
SPQ4σεC	Solid plate quadrilateral based on the consistent assumed stress-strain ANS formulation (section 4.1)
SPQ4σεE	Solid plate quadrilateral based on the enhanced assumed stress-strain ANS formulation (section 4.2)
SPQ4	Solid plate quadrilateral based on the displacement-based ANS formulation (section 3)

5.1 Cantilever beam under tip load

A cantilever beam of the rectangular cross section is subjected to four concentrated loads acting on the bottom and top planes of the free end. Its mechanical and geometrical characteristics are given in Fig. 3. The six element mesh is used to model this problem and all elements are distorted so as to keep their widths on the centerline constant.

The normalized transverse tip displacements of the centerline $\left(u_3^{\text{M}}\right)^{\text{Norm}}$ are displayed in Table 2. The displacements are normalized with respect to the analytical solution [17]. It is seen that both developed assumed stress-strain elements perform excellently.



$\ell = 3.6$, $d = 0.95$, $h = 0.036$, $E = 2\times10^6$, $\nu = 0$, $F = 100$

Fig. 3 Cantilever beam under tip load

Table 2

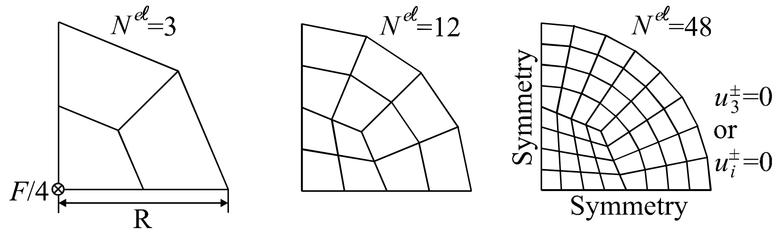
Normalized transverse tip displacement $\left(u_3^{\mathrm{M}}\right)^{\mathrm{Norm}}$ of cantilever beam

a/b	SPQ4σεC	SPQ4σεE	SPQ4	MITC4 [17]
1	0.9931	0.9931	0.9931	0.9931
5	0.9867	0.9879	0.9867	0.9825
59	0.9817	0.9848	0.9825	0.9704

5.2 Circular plate under central load

Consider a thin circular plate subjected to a concentrated load at the center point. The mechanical and geometrical characteristics of the plate are given in Fig. 4. The standard meshes [6] are used to model due to symmetry only one quarter of the plate.

Fig. 5 displays transverse central midplane displacements u_3^{M} of the simply supported and clamped plates normalized with respect to the analytical solution [18], based on the classical plate theory. Additionally, a comparison with results of Hughes and Tezduyar [6] is presented. The predictions of SPQ4σεE and SPQ4 elements are graphically indistinguishable from that of SPQ4σεC and T1 [6] ones, respectively, and are not shown in Fig. 5(b).



R=5, h=0.1, E=1.092×10^6, ν=0.3, F=1

Fig. 4 Circular plate under central load

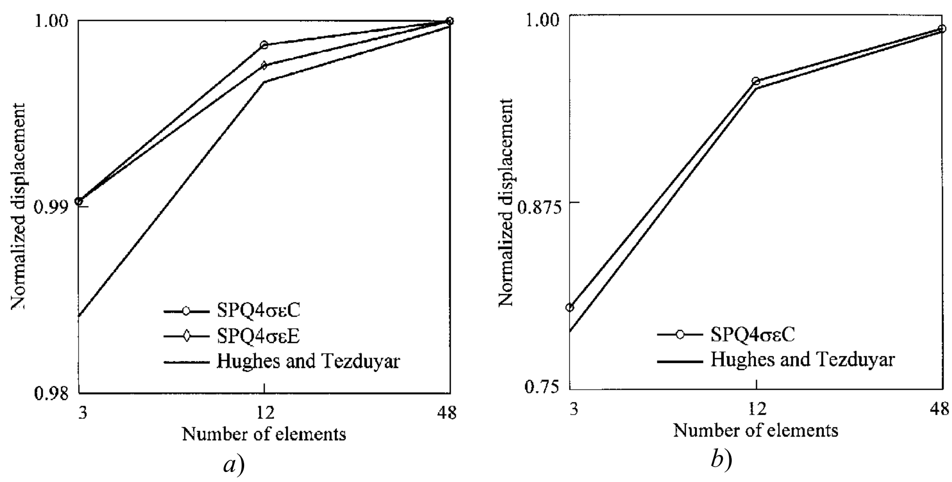


Fig. 5 Convergence study due to mesh refinement for (*a*) simply supported and (*b*) clamped circular plates

8 Conclusions

On the basis of the first-order solid plate theory the assumed stress-strain ANS quadrilateral solid plate elements have been developed. Because of locking in the case of coarse skewed meshes in several test problems in addition to the consistent assumed stress-strain ANS formulation it has been presented the more robust one. We refer to it as an enhanced assumed stress-strain ANS formulation. Both assumed stress-strain quadrilaterals permit the analytical integration leading to the elemental stiffness matrices because fundamental matrices $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, corresponding to each finite element formulation, possess very simple structures and their analytical inverse can be carried out in a closed form.

It is important that element stiffness matrices of elaborated assumed stress-strain quadrilateral four-node elements on the basis of the analytical and numerical integration have six, and only six, zero eigenvalues as required for satisfaction of the general rigid-body motion representation. Besides, CPU time required for the formation of the global stiffness matrix is slightly depend on a number of elements used for discretizing a plate and the numerically integrated element matrix needs about 50 % more time than the analytically integrated one.

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References

1 Pian, T.H.H. Derivation of element stiffness matrices by assumed stress distributions / T.H.H. Pian // AIAA Journal. – 1964. – Vol. 2. – Pp. 1333-1336.

2 Pian, T.H.H. State-of-the-art development of hybrid/mixed finite element method / T.H.H. Pian // Finite Elements in Analysis and Design. – 1995. – Vol. 21. – Pp. 5-20.

3 Wempner, G. A simple and efficient approximation of shells via finite quadrilateral elements / G. Wempner, D. Talaslidis, C.M. Hwang // Trans. ASME, Journal of Applied Mechanics. – 1982. – Vol. 49. – Pp. 115-120.

4 Rhiu, J.J. A new efficient mixed formulation for thin shell finite element models / J.J. Rhiu, S.W. Lee // International Journal for Numerical Methods in Engineering. – 1987. – Vol. 24. – Pp. 581-604.

5 Sze, K.Y. Three-dimensional continuum finite element models for plate/shell analysis / K.Y. Sze // Progress in Structural Engineering and Materials. – 2002. –Vol. 4. – Pp. 400-407.

6 Hughes, T.J.R. Finite elements based upon Mindlin plate theory with particular reference to the four-node bilinear isoparametric element / T.J.R. Hughes, T.E. Tezduyar // Trans. ASME, Journal of Applied Mechanics. – 1981. – Vol. 48. – Pp. 587-596.

7 Simo, J.C. On a stress resultant geometrically exact shell model. Part IV. Variable thickness shells with through-the-thickness stretching / J.C. Simo, M.S. Rifai, D.D. Fox // Computer Methods in Applied Mechanics and Engineering. – 1990. – Vol. 81. – Pp. 91-126.

8 Sze, K.Y. An eight-node hybrid-stress solid-shell element for geometric nonlinear analysis of elastic shells / K.Y. Sze, W.K. Chan, T.H.H. Pian // International Journal for Numerical Methods in Engineering. – 2002. – Vol. 55. – Pp. 853-878.

9 Park, H.C. An efficient assumed strain element model with six dof per node for geometrically nonlinear shells / H.C. Park, C. Cho, S.W. Lee // International Journal for Numerical Methods in Engineering. – 1995. – Vol. 38. – Pp. 4101-4122.

10 Kulikov, G.M. Analysis of initially stressed multilayered shells / G.M. Kulikov // International Journal of Solids and Structures. – 2001. – Vol. 38. – Pp. 4535-4555.

11 Kulikov, G.M. Simple and effective elements based upon Timoshenko-Mindlin shell theory / G.M. Kulikov, S.V. Plotnikova // Computer Methods in Applied Mechanics and Engineering. – 2002. – Vol. 191. – Pp. 1173-1187.

12 Kulikov, G.M. Non-linear strain-displacement equations exactly representing large rigid-body motions. Part I. Timoshenko-Mindlin shell theory / G.M. Kulikov, S.V. Plotnikova // Computer Methods in Applied Mechanics and Engineering. – 2003. – Vol. 192. – Pp. 851-875.

13 Kulikov, G.M. Finite deformation plate theory and large rigid-body motions / G.M. Kulikov, S.V. Plotnikova // International Journal of Non-Linear Mechanics. – 2004. – Vol. 39. – Pp. 1093-1109.

14 Kulikov, G.M. Equivalent single-layer and layer-wise shell theories and rigid-body motions. Part I. Foundations / G.M. Kulikov, S.V. Plotnikova // Mechanics of Advanced Materials and Structures. – 2005. – Vol. 12. – Pp. 275-283.

15 Pian, T.H.H. Rational approach for assumed stress finite elements / T.H.H. Pian, K. Sumihara // International Journal for Numerical Methods in Engineering. – 1984. – Vol. 20. – Pp. 1685-1695.

16 Kulikov, G.M. Equivalent single-layer and layer-wise shell theories and rigid-body motions. Part II. Computational aspects / G.M. Kulikov, S.V. Plotnikova // Mechanics of Advanced Materials and Structures. – 2005. – Vol. 12. – Pp. 331-340.

17 Bathe, K.J. A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation / K.J. Bathe, E.N. Dvorkin // International Journal for Numerical Methods in Engineering. – 1985. – Vol. 21. – Pp. 367-383.

18 Timoshenko, S.P. Theory of Plates and Shells, 2nd ed. / S.P. Timoshenko, S. Woinowsky-Krieger. – New York: McGraw-Hill, 1970.

Четырехугольные элементы пластины с введенным распределением напряжений и деформаций на основе аналитического и численного интегрирования

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Ключевые слова и фразы: метод введенных локальных деформаций (ANS метод); смешанные конечно-элементные модели; четырехузловой трехмерный элемент пластины.

Аннотация: На основе трехмерной теории пластин первого порядка, предложенной ранее авторами, построены два четырехугольных ANS элемента с введенным распределением напряжений и деформаций. Настоящая статья обобщает общепринятый подход к построению трехмерного смешанного элемента для анализа пластины и оболочки, так как поперечные касательные деформации распределены по толщине согласно линейному закону. Предложенные трехмерные ANS элементы пластины основаны на общей методологии, что позволяет оценить

их достоинства и недостатки. Примечательно, что матрицы жесткости разработанных четырехугольных элементов, включая четырехузловой ANS элемент на основе метода перемещений, обладают шестью нулевыми собственными значениями, что необходимо для представления движения элемента как твердого тела.

Viereckige Elemente der Platte mit der eingeführten Einteilung der Spannungen und der Entstellungen auf Grund der analytischen und numerischen Integration

Zusammenfassung: Auf Grund der dreidimensionalen Theorie der Platten der ersten Ordnung, die früher von den Autoren angeboten ist, sind zwei viereckige ANS-Elemente mit der eingeführten Einteilung der Spannungen und der Entstellungen aufgebaut. Der vorliegende Artikel fasst das allgemeingültige Herangehen zur Konstruktion des dreidimensionalen gemischten Elementes für die Analyse der Platte und die Hülle zusammen, da die querlaufenden Tangenten der Entstellung nach der Dicke laut des linearen Gesetzes verteilt sind. Die angebotenen dreidimensional ANS sind die Elemente der Platte auf der gemeinen Methodologie gegründet, daß ihre Vorteile und Nachteile zu schätzen erlaubt. Es ist bemerkenswert, daß die Härtematrizen der entwickelten viereckigen Elemente, einschließlich das viernodösen ANS-Element aufgrund der Umstellungsmethode, über sechs Null-eigenen Bedeutungen haben, was für die Darstellung der Bewegung des Elementes als des festen Körpers notwendig ist.

Eléments quadrangles de la plaquette avec la répartition introduite des tensions et des déformations à la base de l'intégration analytique et numérique

Résumé: A la base de la théorie des plaquettes tridimensionnelles de première ordre proposée par les auteurs auparavant, sont construits deux éléments quadrangles ANS avec la répartition introduite des tensions et des déformations. Le présent article généralise une approche admise par tous envers la construction d'un élément mixte tridimensionnel pour l'analyse de la plaquette et de l'enveloppe puisque les déformations transversales et tangentielles sont réparties par l'épaisseur selon la loi linéaire. Les éléments ANS de la plaquette proposés sont fondés sur la méthode commune ce qui permet d'évaluer leurs avantages et leurs défauts. Il est important que les matrices de la solidité des éléments quadrangles élaborés y compris l'élément ANS à quatre noeuds possèdent six significations propres «zéro» ce qui est nécessaire pour la présentation du mouvement d'un élément comme corp solide.
